

Degasperis–Procesi peakons and the discrete cubic string

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March 15, 2005

Abstract

We use an inverse scattering approach to study multi-peakon solutions of the Degasperis–Procesi (DP) equation, an integrable PDE similar to the Camassa–Holm shallow water equation. The spectral problem associated to the DP equation is equivalent under a change of variables to what we call the cubic string problem, which is a third order non-selfadjoint generalization of the well-known equation describing the vibrational modes of an inhomogeneous string attached at its ends. We give two proofs that the eigenvalues of the cubic string are positive and simple; one using scattering properties of DP peakons, and another using the Gantmacher–Krein theory of oscillatory kernels.

For the discrete cubic string (analogous to a string consisting of n point masses) we solve explicitly the inverse spectral problem of reconstructing the mass distribution from suitable spectral data, and this leads to explicit formulas for the general n -peakon solution of the DP equation. Central to our study of the inverse problem is a peculiar type of simultaneous rational approximation of the two Weyl functions of the cubic string, similar to classical Padé–Hermite approximation but with lower order of approximation and an additional symmetry condition instead.

The results obtained are intriguing and nontrivial generalizations of classical facts from the theory of Stieltjes continued fractions and orthogonal polynomials.

Keywords: Inverse problem, peakons, Weyl function, cubic string, Padé approximation.

MSC2000 classification: 35Q51, 34K29, 37J35, 35Q53, 34B05, 41A21.

To appear in *International Mathematics Research Papers*.

1 Introduction

The PDE

$$u_t - u_{xxt} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx}, \quad (x, t) \in \mathbf{R}^2, \quad (1.1)$$

is known to be completely integrable if and only if $b = 2$ or $b = 3$. The case $b = 2$ is the well-known Camassa–Holm (CH) shallow water equation [4]. The case $b = 3$ is the Degasperis–Procesi (DP) equation, found by Degasperis and Procesi [7] to pass the necessary (but not sufficient) test of asymptotic integrability, and recently shown by Degasperis, Holm and Hone [5, 6] to be integrable indeed.

All equations in the family (1.1) admit (in a weak sense) a type of non-smooth solutions called *multi-peakons* (peakon = peaked soliton). These take the form of a train of peak-shaped interacting waves,

$$u(x, t) = \sum_{i=1}^n m_i(t) e^{-|x-x_i(t)|}, \quad (1.2)$$

where the positions $x_i(t)$ and the heights $m_i(t)$ are determined by the following system of nonlinear ODEs:

$$\begin{aligned} \dot{x}_k &= \sum_{i=1}^n m_i e^{-|x_k - x_i|}, \\ \dot{m}_k &= (b-1) \sum_{i=1}^n m_k m_i \operatorname{sgn}(x_k - x_i) e^{-|x_k - x_i|}, \end{aligned} \quad (1.3)$$

for $k = 1, \dots, n$. Here we use the convention that $\operatorname{sgn} 0 = 0$, and dots denote $\frac{d}{dt}$ as usual. Throughout the paper, n will be fixed but arbitrary.

In the integrable CH and DP cases, explicit solution formulas for $x_i(t)$ and $m_i(t)$ can be found using inverse scattering techniques, which makes it possible to analyze the peakon interactions in great detail. This was shown for the CH case $b = 2$ by Beals, Sattinger and Szmigielski [3, 2]. The DP case $b = 3$ was briefly treated in a short note by the present authors [15], where we outlined a recursive procedure to determine solution formulas for any given n . It is our purpose here to continue this study in more detail. Among other things, we will give the promised closed form n -peakon solution of the DP equation (see Corollary 2.23).

This paper is divided into two major parts. In Section 2 we study the peakon solutions of the DP equation. First we prove certain scattering properties of the peakons, using only elementary methods and the governing equations themselves. Then we describe the inverse scattering procedure, based on the Lax pair given in [5]. The spectral problem associated to the DP equation is of third order and non-selfadjoint. To prove that the spectrum is nevertheless real and simple, as well as some other related facts that we need, we make extensive use of the scattering properties derived earlier. We state the resulting formulas for the general n -peakon solution, and extract precise asymptotic properties as $t \rightarrow \pm\infty$.

Sections 3 and 4 are devoted to the forward and inverse spectral problems for the *discrete cubic string*, our main tool for proving the explicit n -peakon formulas, but also a very interesting problem in its own right from the point of view of operator theory. By the forward cubic string problem, we mean the following third order spectral problem: for a given function $g(y) \geq 0$, determine the eigenvalues z such that nontrivial eigenfunctions $\phi(y)$ exist satisfying

$$\begin{aligned} -\phi_{yyy}(y) &= z g(y) \phi(y) \quad \text{for } y \in (-1, 1), \\ \phi(-1) &= \phi_y(-1) = 0, \quad \phi(1) = 0. \end{aligned}$$

This spectral problem, which is equivalent under a change of variables to the one appearing in the DP Lax pair, is a non-selfadjoint generalization of the well-known (selfadjoint) string equation

$$\begin{aligned} \phi_{yy}(y) &= z g(y) \phi(y) \quad \text{for } y \in (-1, 1), \\ \phi(-1) &= 0, \quad \phi(1) = 0, \end{aligned}$$

describing the vibrational modes of a string with nonhomogeneous mass density $g(y)$, attached at the ends $y = \pm 1$. The discrete case arises when $g = \sum_{i=1}^n g_i \delta_{y_i}$ is a discrete measure (“point masses g_i at the positions y_i ”), so that the eigenfunctions are piecewise linear in y for the ordinary string and piecewise quadratic for the cubic string.

From what we proved about peakon scattering, it follows that the eigenvalues of the discrete cubic string are positive and simple. We also provide an independent proof of this, valid not only in the discrete case, using the theory of oscillatory kernels developed by Gantmacher and Krein [12].

The discrete (ordinary) string plays a crucial role in finding the general n -peakon solution for the CH equation [3, 2]. The inverse spectral problem consists of determining the positions y_i and masses g_i given the eigenfrequencies and suitable additional information about the eigenfunctions (encoded in the *spectral measure* of the string, or equivalently in its Stieltjes transform, henceforth called the *Weyl function*). The solution, found by Krein (and outlined in Appendix A for the reader’s convenience), involves Stieltjes continued fractions, a subject with extremely rich connections to various mathematical areas such as the classical moment problem, Padé approximation, and orthogonal polynomials.

The corresponding inverse problem for the discrete cubic string, which we solve in Section 4, leads to new and very intriguing generalizations of much of this. There are two Weyl function, one of which is unexpectedly determined by the other (the proof of this “static” fact uses the isospectral deformation of the cubic string induced by the DP dynamics). These two Weyl functions admit simultaneous approximation by rational functions with a common denominator, similar to classical Padé–Hermite approximation of two (independently chosen) functions, but with lower order of approximation and instead an additional symmetry condition, and also similar to the Padé approximation provided by the convergents of the Stieltjes continued fraction in the ordinary string case. The study of this new type of approximation problem is the key to the solution of the inverse spectral problem. The coefficients in the common denominator of the rational approximants are given by ratios of determinants involving certain moment-like *double* integrals of a spectral measure. The evaluation of these determinants is significantly more difficult than for the classical case, where the well-known Hankel determinants of moments of the spectral measure appear instead.

It may be expected that Krein’s solution of the inverse spectral problem in the case of a string with a general mass distribution also admits a generalization to the cubic string, but we have not endeavored to address that question here.

2 Peakon solutions of the DP equation

2.1 The governing equations

We begin by summarizing some basic facts about the Degasperis–Procesi (DP) equation

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}. \quad (2.1)$$

It is advantageous to write it as a system,

$$0 = m_t + m_x u + 3m u_x, \quad (2.2a)$$

$$m = u - u_{xx}. \quad (2.2b)$$

The function $m(x, t)$ can be thought of as a momentum-like quantity, although we are not aware of any actual physical interpretation of the DP equation. The n -peakon solution

$$u(x, t) = \sum_{i=1}^n m_i(t) e^{-|x - x_i(t)|} \quad (2.3)$$

arises when m is not a function but a finite discrete measure,

$$m(x, t) = 2 \sum_{i=1}^n m_i(t) \delta(x - x_i(t)). \quad (2.4)$$

Then (2.2b) is satisfied by construction, while (2.2a) is satisfied in a weak sense if and only if the positions $x_1(t), \dots, x_n(t)$ and the heights $m_1(t), \dots, m_n(t)$

satisfy the following system, which was given in [5] and is one of our fundamental objects of study here:

$$\begin{aligned}\dot{x}_k &= \sum_{i=1}^n m_i e^{-|x_k - x_i|}, \\ \dot{m}_k &= 2 \sum_{i=1}^n m_k m_i \operatorname{sgn}(x_k - x_i) e^{-|x_k - x_i|},\end{aligned}\tag{2.5}$$

for $k = 1, \dots, n$.

2.2 Examples of explicit solutions

Finding the solution of (2.5) in the case $n = 1$ is trivial: $m_1 = \text{constant}$ and $x_1 = x_1(0) + m_1 t$. We can write this in a way that anticipates the general pattern:

$$m_1(t) = \frac{b_1^2}{\lambda_1 b_1^2}, \quad x_1(t) = \log b_1,\tag{2.6}$$

where $b_1(t) = b_1(0)e^{t/\lambda_1}$, and the constants $\lambda_1 = 1/m_1(0)$ and $b_1(0) = e^{x_1(0)}$ are determined by initial conditions.

Also when $n = 2$, the equations can be integrated by straightforward methods if one changes to variables $x_1 \pm x_2$ and $m_1 \pm m_2$ as discussed in [5]. In our notation, the solution takes the form

$$\begin{aligned}x_1(t) &= \log \frac{\frac{(\lambda_1 - \lambda_2)^2}{\lambda_1 + \lambda_2} b_1 b_2}{\lambda_1 b_1 + \lambda_2 b_2}, \\ x_2(t) &= \log(b_1 + b_2), \\ m_1(t) &= \frac{(\lambda_1 b_1 + \lambda_2 b_2)^2}{\lambda_1 \lambda_2 \left(\lambda_1 b_1^2 + \lambda_2 b_2^2 + \frac{4\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} b_1 b_2 \right)}, \\ m_2(t) &= \frac{(b_1 + b_2)^2}{\lambda_1 b_1^2 + \lambda_2 b_2^2 + \frac{4\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} b_1 b_2},\end{aligned}\tag{2.7}$$

where $b_k(t) = b_k(0)e^{t/\lambda_k}$, and the constants $\lambda_1, \lambda_2, b_1(0), b_2(0)$ are determined by initial conditions.

For $n \geq 3$ the peakon equations become considerably more involved, and so also the explicit solution formulas, if written out in full. However, if one introduces suitable notation for certain symmetric functions of b_k 's and λ_k 's, the formulas become very succinct; see Definitions 2.18 and 2.19 for notation, Corollary 2.23 for the general statement and Example 2.24 for the three-peakon solution written out in detail.

Remark 2.1. To write the formulas for $x_1(t)$ and $x_2(t)$ in (2.7) in the form

given in [5], substitute

$$\lambda_1 = \frac{1}{c_1}, \quad \lambda_2 = \frac{1}{c_2}, \quad P = c_1 + c_2,$$

$$b_1(0) = \frac{\sqrt{\Gamma}}{2(c_1 - c_2)}, \quad b_2(0) = \frac{2c_2P}{(c_1 - c_2)\sqrt{\Gamma}}.$$

There is a small error in [5]: the term $4c_2P$ in their equation (6.8) should be $4c_1P$, and vice versa in (6.9). However, their phase shift formulas (6.10) and (6.11) are correct, and consistent with the corrected (6.8) and (6.9).

2.3 A priori estimates

In this section we derive some useful information about the peakon dynamics directly from the governing equations (2.5). Although this is instructive in itself, our primary reason for doing this is that we will use this information in order to show that certain eigenvalues λ_k that appear later in the inverse scattering analysis are real and simple.

There is a distinction between peakons ($m_k > 0$) and antipeakons ($m_k < 0$). As indicated by the formulas for $n = 1$ and $n = 2$ above, peakons move to the right and antipeakons to the left. If both are present, the system (2.5) blows up after a finite time when the first collision between a peakon and an antipeakon occurs (their corresponding m_k 's diverge to $+\infty$ and $-\infty$, respectively). In the case of the CH equation, $u(x, t)$ can nevertheless be continued in a natural way beyond the time of collision [2]. The behaviour of the DP equation seems to be more complicated in this respect. In this paper we will concentrate on the pure peakon case (all $m_k > 0$), except for Section 2.10 where we will make a few comments about antipeakons.

Assumption 2.2. Except where otherwise stated, we assume the following:

1. Pure peakons: $m_k(t) > 0$ for $k = 1, \dots, n$.
2. Ordering: $x_1(t) < x_2(t) < \dots < x_n(t)$.

We call the set where Assumption 2.2 holds the “pure peakon sector” of \mathbf{R}^{2n} and we denote it by \mathcal{P} . In other words,

$$\mathcal{P} = \{(x, m) \in \mathbf{R}^{2n} : x_1 < \dots < x_n, \text{ all } m_i > 0\}. \quad (2.8)$$

For later convenience, we also introduce the definitions

$$x_0 = -\infty \quad \text{and} \quad x_{n+1} = \infty, \quad (2.9)$$

with natural interpretations like $e^{x_0} = 0$ and $\tanh x_0 = -1$.

Proposition 2.3 (Global pure peakon solution). *Given initial conditions consistent with Assumption 2.2 at some time $t = t_0$, there exists a unique solution to (2.5) defined for all $t \in \mathbf{R}$ and satisfying these initial conditions. There*

are constants C and D such that $m_k(t) > C > 0$ and $x_{k+1}(t) - x_k(t) > D > 0$ for all k and all $t \in \mathbf{R}$. In particular, the solution remains in the interior of \mathcal{P} for all $t \in \mathbf{R}$.

Proof. The right-hand side of (2.5) is clearly uniformly Lipschitz continuous as a function of x_k 's and m_k 's on any compact subset of \mathcal{P} , and thus the local existence and uniqueness follows from the Picard–Lindelöf theorem. Moreover, since the right-hand side of (2.5) is smooth on \mathcal{P} , the solution is smooth on the domain of its existence. To show global existence, we need to rule out for the solution to: (1) intersect in finite time the boundary of \mathcal{P} , (2) blow up in finite time.

It can be verified directly from the equations (2.5) that $M_1 = m_1 + \dots + m_n$ and $M_n = \left(\prod_{k=1}^n m_k\right) \left(\prod_{k=1}^{n-1} (1 - e^{x_k - x_{k+1}})^2\right)$ are constants of motion as long as the ordering assumption is satisfied, which is at least true for all t in some open interval T containing t_0 . (In Theorem 2.10 we show that M_1 and M_n belong to a set of n constants of motion M_1, \dots, M_n .)

The pure peakons assumption implies that M_1 and M_n are positive, and that $m_k(t) < M_1$ for all k and all $t \in T$. Using this estimate on all but one factor m_k in M_n , and noting that each factor in the second product is ≤ 1 due to the ordering assumption, we get $M_n < m_k(t) M_1^{n-1} 1^{n-1}$. Hence $m_k(t) > M_n / M_1^{n-1} > 0$ for all k and all $t \in T$, so the pure peakons assumption holds in T . Since by definition of T the ordering assumption holds there, we conclude that both the pure peakon and the ordering assumption hold in T . Similarly, $(1 - e^{x_k - x_{k+1}})^2 > M_n / (M_1^n 1^{n-2}) =: c \in (0, 1)$, hence $x_{k+1}(t) - x_k(t) > -\log(1 - c) > 0$ for each k . Since the distances between peakons are bounded from below by a strictly positive constant, the peakons cannot change places. Thus for no finite time can peakons reach the boundary of \mathcal{P} .

Finally, no blow-up of solutions in finite time is admitted because by the second equation (2.5), and with the help of $0 < M_n / M_1^{n-1} < m_k < M_1$, we get $M_n / M_1^{n-1} < \dot{x}_j < M_1$. Thus $(M_n / M_1^{n-1})t < x_j - x_j(0) < M_1 t$, implying that the solution stays bounded at any finite time and the ordering assumption holds for all t (that is, $T = \mathbf{R}$). \square

By direct analysis of (2.5) we obtain the following theorem which shows that the peakons scatter and asymptotically behave like free particles travelling with distinct velocities. This behaviour in mechanical systems, integrable by the Lax method, was first observed and analyzed by J. Moser in [16].

The results of the next theorem will be superseded by more precise ones in Section 2.9 when we have at our disposal the explicit formulas for the n -peakon solution. However, we will use these *a priori* estimates in order to show that certain eigenvalues $\lambda_1, \dots, \lambda_n$ appearing in the solution formulas must be real and distinct.

Theorem 2.4 (Asymptotics). *Let $\{x_k(t), m_k(t)\}$ be a solution to (2.5) satisfying Assumption 2.2. Then the following claims are true:*

1. *The limits $m_k(\pm\infty) := \lim_{t \rightarrow \pm\infty} m_k(t)$ exist and are finite.*

2. Peakons travel to the right and scatter; i.e., $\dot{x}_k(t) > 0$, and $x_{k+1}(t) - x_k(t) \rightarrow \infty$ as $t \rightarrow \pm\infty$.
3. $\lim_{t \rightarrow \pm\infty} \dot{x}_k(t) = m_k(\pm\infty)$, and $x_k(t) = m_k(\pm\infty)t + O(1)$ as $t \rightarrow \pm\infty$.
4. $0 < m_1(\infty) < m_2(\infty) < \dots < m_n(\infty)$ and $0 < m_n(-\infty) < \dots < m_2(-\infty) < m_1(-\infty)$.

Proof. We will only prove the statements concerning $t \rightarrow +\infty$, as the case $t \rightarrow -\infty$ is completely analogous. To improve readability, let $d_{ij} = |x_i - x_j|$ denote the distance between the i th peakon and the j th. If we impose the restriction that the smaller index always be written first ($i < j$), we can remove the absolute value signs by Assumption 2.2 and Proposition 2.3: $d_{ij} = x_j - x_i > 0$. Then the peakon equations (2.5) read

$$\begin{aligned}\dot{x}_k &= \left(\sum_{i < k} m_i e^{-d_{ik}} \right) + m_k + \left(\sum_{i > k} m_i e^{-d_{ki}} \right), \\ \frac{\dot{m}_k}{2m_k} &= \left(\sum_{i < k} m_i e^{-d_{ik}} \right) - \left(\sum_{i > k} m_i e^{-d_{ki}} \right).\end{aligned}\tag{2.10}$$

We prove claims 1 and 2 together. To begin with, it follows from (2.10) that $\dot{x}_k > 0$ since all $m_k > 0$. Now we proceed by induction on k , starting with n and descending to 1. The equation for m_n is

$$\frac{\dot{m}_n}{2m_n} = \sum_{i=1}^{n-1} m_i e^{-d_{in}} > 0,$$

so m_n is increasing, and since m_n is bounded from above by the constant of motion $M_1 = \sum_1^n m_i$, it follows that $\lim_{t \rightarrow \infty} m_n$ exists and is finite. Hence the integral in

$$m_n(t) = m_n(0) \exp \left(\int_0^t 2 \sum_{i=1}^{n-1} m_i(s) e^{-d_{in}(s)} ds \right),$$

is convergent. Since the m_i 's are positive and bounded away from zero, this implies that

$$\int_0^\infty e^{-d_{in}(s)} ds < \infty, \quad 1 \leq i \leq n-1.$$

The derivative of the integrand $e^{-d_{in}(t)}$ is bounded: $|\dot{x}_k| \leq \sum_1^n m_i = M_1$ by (2.10), so

$$\left| \frac{d}{dt} e^{-d_{in}(t)} \right| = |(\dot{x}_n - \dot{x}_i) e^{-d_{in}}| \leq |\dot{x}_n| + |\dot{x}_i| \leq 2M_1.$$

This means that $e^{-d_{in}(t)}$ must vanish as $t \rightarrow \infty$, otherwise the integral would be divergent. (Suppose $u(t) > 0$, $|\dot{u}(t)| \leq C$, and u does *not* vanish as $t \rightarrow \infty$; say

$\limsup_{t \rightarrow \infty} u(t) = \epsilon > 0$. Then there is an increasing sequence of points $t_j \rightarrow \infty$ (which can be taken to lie more than $\epsilon/4C$ apart) such that $u(t_j) > \epsilon/2$, and hence $u(t) > \epsilon/4$ in the interval $[t_j, t_j + \epsilon/4C]$ because of the bound on \dot{u} . This forces $\int_0^\infty u(s) ds = \infty$.) Consequently, the n th peakon scatters away from the ones to its left:

$$\lim_{t \rightarrow \infty} d_{in}(t) = \infty, \quad i = 1, \dots, n-1. \quad (2.11)$$

We have now proved the base step of the induction. When considering the k th peakon, we make the following induction hypothesis about the peakons to its right: for all $j > k$,

- i. $m_j(\infty) = \lim_{t \rightarrow \infty} m_j(t)$ exists and is finite.
- ii. The j th peakon scatters away from the ones to its left: $\int_0^\infty e^{-d_{ij}(s)} ds < \infty$ and $\lim_{t \rightarrow \infty} d_{ij}(t) = \infty$ for $i = 1, \dots, j-1$.

We need to show that the same holds also for $j = k$. Integration of the equation for m_k in (2.10) yields

$$\begin{aligned} m_k(t) \exp \left(2 \int_0^t \sum_{j>k} m_j(s) e^{-d_{kj}(s)} ds \right) &= \\ &= m_k(0) \exp \left(2 \int_0^t \sum_{i<k} m_i(s) e^{-d_{ik}(s)} ds \right). \end{aligned}$$

Since the right-hand side is an increasing function of t , we know that its limit exists, even though it might be infinite. By the induction hypothesis the integral on the left-hand side has a finite limit. We conclude that $m_k(\infty) = \lim_{t \rightarrow \infty} m_k(t)$ exists, and this limit is finite because m_k is bounded. Hence the limit of the right-hand side is in fact finite, which, using the same arguments as in the base step of the induction, implies $\int_0^\infty e^{-d_{ik}(s)} ds < \infty$ and $\lim_{t \rightarrow \infty} d_{ik}(t) = \infty$ for $i = 1, \dots, k-1$; that is, the k th peakon scatters away from the ones to its left. This concludes the proof of claims 1 and 2.

The scattering property just proved implies that all terms in the equation for \dot{x}_k in (2.10) vanish as $t \rightarrow \infty$, except the lone m_k . Consequently, by l'Hospital's rule, $\lim_{t \rightarrow \infty} x_k(t)/t = \lim_{t \rightarrow \infty} \dot{x}_k/1 = m_k(\infty)$, which doesn't quite prove claim 3 yet, since at this stage we can only draw the following weaker conclusion:

$$x_k(t) = m_k(\infty) t + o(t) \quad \text{as } t \rightarrow \infty. \quad (2.12)$$

We will improve the remainder to $O(1)$ after using (2.12) to prove claim 4.

Proposition 2.3 tells us that m_1 is bounded away from zero, hence $0 < m_1(\infty)$, and that the ordering $x_1 < \dots < x_n$ is preserved, hence (in order not to contradict (2.12))

$$m_1(\infty) \leq m_2(\infty) \leq \dots \leq m_n(\infty).$$

To prove claim 4, it remains to show that these inequalities are actually strict. Suppose, to derive a contradiction, that not all $m_k(\infty)$ are distinct. Then there is a sequence of two or more consecutive $m_k(\infty)$ that are equal, say

$$m_{a-1}(\infty) < m_a(\infty) = \cdots = m_b(\infty) < m_{b+1}(\infty) \quad (2.13)$$

for some $a < b$. (If $a = 1$, then $m_{a-1}(\infty)$ is not present here, and similarly $m_{b+1}(\infty)$ is not present if $b = n$.) We will show that this implies the existence of a t_0 such that $\dot{m}_a(t) < 0$ and $\dot{m}_b(t) > 0$ for all $t \geq t_0$. Then we will show that one can find $t_1 > t_0$ such that $m_a(t_1) < m_b(t_1)$. Since m_a will decrease and m_b will increase as t grows, it follows that $m_a(\infty) < m_b(\infty)$, which is the desired contradiction.

The case $a = 1$ is trivial, since it is clear from (2.10) that $m_1(t)$ is decreasing for all t . So assume $2 \leq a < n$ and rewrite the equation for m_a by factoring out the exponential in the last positive term ($i = a - 1$):

$$\frac{\dot{m}_a}{2m_a} = e^{-d_{a-1,a}} \left[\left(\sum_{i=1}^{a-2} m_i e^{-d_{i,a-1}} \right) + m_{a-1} - \left(\sum_{i=a+1}^n m_i e^{-d_{a,i}+d_{a-1,a}} \right) \right].$$

Under the assumption (2.13), equation (2.12) implies that $d_{a-1,a} = ct + o(t)$ with $c = m_a(\infty) - m_{a-1}(\infty) > 0$, and $d_{a,a+1} = o(t)$ (since $m_a(\infty) = m_{a+1}(\infty)$). Hence the exponent in the term corresponding to $i = a + 1$ is $ct + o(t)$, which, since the factor m_{a+1} in front of the exponential is bounded away from zero, means that the second sum tends to infinity as $t \rightarrow \infty$. The first sum tends to zero by scattering (unless $a = 2$ in which case it is empty), and m_{a-1} is bounded. Consequently, the right-hand side is negative for all sufficiently large t , so there is a t_a such that $m_a(t)$ is decreasing for $t \geq t_a$.

Similarly, $m_n(t)$ is increasing so $b = n$ is a trivial case. If $1 < b \leq n - 1$, rewrite the equation for m_b by factoring out the exponential in the first negative term ($i = b + 1$):

$$\frac{\dot{m}_b}{2m_b} = e^{-d_{b,b+1}} \left[\left(\sum_{i=1}^{b-1} m_i e^{-d_{ib}+d_{b,b+1}} \right) - m_{b+1} - \left(\sum_{i=b+2}^n m_i e^{-d_{b+1,i}} \right) \right].$$

The right-hand side is positive for large t , since the second sum is either empty (if $b = n - 1$) or tends to zero by scattering, m_{b+1} is bounded, and the term corresponding to $i = b - 1$ in the first sum tends to infinity because its exponent is $ct + o(t)$ with $c = m_{b+1}(\infty) - m_b(\infty) > 0$. Hence $m_b(t)$ is increasing for $t \geq t_b$, say. Now simply let $t_0 = \max(t_a, t_b)$. Then $\dot{m}_a(t) < 0$ and $\dot{m}_b(t) > 0$ for all $t \geq t_0$, as was to be shown.

Finally, observe that

$$\dot{d}_{ab} = \dot{x}_b - \dot{x}_a = m_b - m_a + \left(\sum_{i \neq b} m_i e^{-|x_i - x_b|} \right) - \left(\sum_{i \neq a} m_i e^{-|x_i - x_a|} \right).$$

The integral of this from $t = 0$ to ∞ diverges, since $d_{ab} \rightarrow \infty$ by claim 2. However, the contributions from the sums in parentheses are finite by the proof of claim 2. This means that

$$\int_0^\infty (m_b(t) - m_a(t)) dt = +\infty,$$

which cannot hold if the integrand is nonpositive for all $t \geq t_0$. This shows that there is a $t_1 > t_0$ such that $m_b(t_1) - m_a(t_1) > 0$, which completes the proof of claim 4.

Now it only remains to finish the proof of claim 3 by sharpening the estimate $o(t)$ in (2.12) to $O(1)$. It will be convenient here to put the absolute value signs back in $d_{ij} = |\dot{x}_i - \dot{x}_j|$ and remove the restriction $i < j$. Let $c_1 = \frac{1}{2} \min_{i \neq j} |m_i(\infty) - m_j(\infty)|$. By claim 4 we know that $c_1 > 0$ and then (2.12) shows that there is a $c_2 > 0$ such that on the interval $t > 0$, say, the uniform estimate

$$m_i(t) e^{-d_{ij}(t)} < c_2 e^{-c_1 t} \quad (i \neq j) \quad (2.14)$$

holds. From the equations of motion (2.10) we get

$$\begin{aligned} x_k(t) - m_k(\infty)t &= x_k(0) + \int_0^t (\dot{x}_k(s) - m_k(\infty)) ds = \\ &= x_k(0) + \left(\sum_{i \neq k} \int_0^t m_i(s) e^{-d_{ik}(s)} ds \right) + \int_0^t (m_k(s) - m_k(\infty)) ds. \end{aligned}$$

The sum in brackets is bounded as $t \rightarrow \infty$, since the integrals are convergent. (We saw this already earlier in this proof, but with the estimate (2.14) we can even see that they converge exponentially fast.) So we need only prove that the last integral is convergent. The expression for $\dot{m}_k/2m_k$ in (2.10) contains $n - 1$ terms, all of the form (2.14), so we obtain

$$\begin{aligned} |m_k(s) - m_k(\infty)| &\leq \int_s^\infty |\dot{m}_k(\xi)| d\xi \leq \\ &\leq \int_s^\infty 2M_1(n-1)c_2 e^{-c_1 \xi} d\xi = 2M_1(n-1) \frac{c_2}{c_1} e^{-c_1 s}, \end{aligned}$$

and the integral of the right-hand side from $s = 0$ to ∞ is convergent. Thus $\int_0^\infty (m_k(s) - m_k(\infty)) ds$ is absolutely convergent, and the proof is finished. \square

2.4 Lax pair

The complete integrability of the DP equation manifests itself in the existence of an associated spectral problem, as is usually the case in soliton theory. It was shown in [5] that the DP equation is the compatibility condition for the

following linear system for the *wavefunction* $\psi(x, t; z)$:

$$(\partial_x - \partial_x^3)\psi = z m\psi, \quad (2.15a)$$

$$\psi_t = [z^{-1}(1 - \partial_x^2) + u_x - u\partial_x] \psi. \quad (2.15b)$$

(The term $z^{-1}(1 - \partial_x^2)$ could be replaced by $z^{-1}(c - \partial_x^2)$ where c is an arbitrary constant, but for our purposes $c = 1$ is the appropriate choice.)

Indeed, with $L = \partial_x^3 - \partial_x + zm$ and $A = z^{-1}(1 - \partial_x^2) + u_x - u\partial_x$, the generalized Lax equation

$$L_t = [A, L] - 3u_x L \quad (2.16)$$

is equivalent to

$$m_t + m_x u + 3mu_x = 0, \quad m_x = (u - u_{xx})_x. \quad (2.17)$$

Clearly (2.17) is satisfied by solutions of the DP equation (2.2). Conversely, any solution of (2.17) such that m and $u - u_{xx}$ vanish as $|x| \rightarrow \infty$ (as is certainly the case for peakons) also satisfies the DP equation.

Out of all possible solutions to (2.15), we distinguish the following one in particular:

Definition 2.5 (DP wavefunction). To a given solution $u(x, t)$, $m(x, t)$ of the DP equation (2.2), and to a given complex number z , we associate the corresponding wavefunction $\psi(x, t; z)$ satisfying (2.15) and the additional asymptotic condition

$$\psi(x, t; z) \sim e^x, \quad \text{as } x \rightarrow -\infty. \quad (2.18)$$

(It is not hard to see that (2.18) is consistent with (2.15) if u and its derivatives decay sufficiently fast as $|x| \rightarrow \infty$; we show the details for the peakon case in Section 2.6 below.)

2.5 The peakon wavefunction $\psi(x, t; z)$

For peakon solutions, when $m = 2 \sum_1^n m_i \delta_{x_i}$ is a discrete measure, the wavefunction ψ of Definition 2.5 can be written down quite explicitly. Since in this case the right-hand side of (2.15a) equals zero away from the points x_i , the wavefunction is a solution of $(\partial_x^3 - \partial_x)\psi = 0$ in each interval (x_k, x_{k+1}) . In other words, ψ is piecewise given by the expression

$$\begin{aligned} \psi(x, t; z) &= A_k(t; z) e^x + B_k(t; z) + C_k(t; z) e^{-x}, \\ x &\in (x_k(t), x_{k+1}(t)) \quad (k = 0, \dots, n). \end{aligned} \quad (2.19)$$

(Recall our convention that $x_0 = -\infty$ and $x_{n+1} = +\infty$.)

In order to see how these pieces fit together, consider for a moment t and z to be fixed and simplify the notation to

$$\psi(x) = A_k e^x + B_k + C_k e^{-x}, \quad x \in (x_k, x_{k+1}). \quad (2.20)$$

According to (2.15a), ψ_{xxx} is proportional to a Dirac delta at each point x_k , which means that ψ and ψ_x are continuous while ψ_{xx} has jump discontinuities:

$$\psi_{xx}(x_k+) - \psi_{xx}(x_k-) = -2zm_k\psi(x_k) \quad (k = 1, \dots, n). \quad (2.21)$$

A straightforward computation translates these continuity and jump conditions into the following relation between the coefficients in adjacent intervals:

$$\begin{pmatrix} A_k \\ B_k \\ C_k \end{pmatrix} = S_k(z) \begin{pmatrix} A_{k-1} \\ B_{k-1} \\ C_{k-1} \end{pmatrix}, \quad (2.22)$$

where the 3×3 matrix S_k (depending on z, x_k, m_k) is given by

$$S_k(z) = I - zm_k \begin{pmatrix} e^{-x_k} \\ -2 \\ e^{x_k} \end{pmatrix} (e^{x_k}, 1, e^{-x_k}). \quad (2.23)$$

Here I denotes the 3×3 identity matrix. Note that $[S_k(z)]^{-1} = S_k(-z)$ and $\det S_k(z) = 1$.

Equation (2.22) determines ψ completely once A_0, B_0 , and C_0 are known. The asymptotic condition (2.18) is in the peakon case actually an equality, $\psi(x) = e^x$ for $x < x_1$, which corresponds to $A_0 = 1$ and $B_0 = C_0 = 0$. So for peakons we get the following special case of Definition 2.5:

Definition 2.6 (DP peakon wavefunction). To a given solution $\{x_k(t), m_k(t)\}_{k=1}^n$ of the DP peakon equations (2.5), and to a given complex number z , we associate the wavefunction $\psi(x, t; z)$ given by (2.19) with coefficients

$$\begin{pmatrix} A_0(t; z) \\ B_0(t; z) \\ C_0(t; z) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad (2.24)$$

$$\begin{pmatrix} A_k(t; z) \\ B_k(t; z) \\ C_k(t; z) \end{pmatrix} = S_k(z) S_{k-1}(z) \cdots S_2(z) S_1(z) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad (k = 1, \dots, n), \quad (2.25)$$

where the dependence on t comes from the dependence of the matrices S_k on $x_k(t)$ and $m_k(t)$.

It is clear from (2.25) and the definition of S_k that A_k, B_k, C_k are polynomials in z of degree k , with coefficients depending on all x_i 's and m_i 's with $i \leq k$, and with constant terms $A_k(t; 0) = 1, B_k(t; 0) = C_k(t; 0) = 0$ (since $S_k(0) = I$). In fact, it is not hard to see that

$$\begin{pmatrix} A_k \\ B_k \\ C_k \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \sum_{p=1}^k \left[\sum_{I \in \binom{[1, k]}{p}} \left(\prod_{i \in I} m_i \right) \left(\prod_{j=1}^{p-1} (1 - e^{x_{i_j} - x_{i_{j+1}}})^2 \right) \begin{pmatrix} 1 \\ -2e^{x_{i_p}} \\ e^{2x_{i_p}} \end{pmatrix} \right] (-z)^p, \quad (2.26)$$

where $\binom{[1,k]}{p}$ is the set of all p -element subsets $I = \{i_1 < \dots < i_p\}$ of the integer interval $[1, k] = \{1, \dots, k\}$. The empty product ($\prod_{j=1}^{p-1}$ in the case $p = 1$) is interpreted as having the value 1.

Example 2.7. From (2.26) one obtains

$$\begin{aligned} A_1 &= 1 - m_1 z, & B_1 &= 2m_1 e^{x_1} z, & C_1 &= -m_1 e^{2x_1} z, \\ A_2 &= 1 - [m_1 + m_2]z + [m_1 m_2 (1 - e^{x_1 - x_2})^2] z^2, \\ B_2 &= 2[m_1 e^{x_1} + m_2 e^{x_2}]z - 2[m_1 m_2 (1 - e^{x_1 - x_2})^2 e^{x_2}] z^2, \\ C_2 &= -[m_1 e^{2x_1} + m_2 e^{2x_2}]z + [m_1 m_2 (1 - e^{x_1 - x_2})^2 e^{2x_2}] z^2, \end{aligned}$$

and

$$\begin{aligned} A_3 &= 1 - [m_1 + m_2 + m_3]z \\ &\quad + [m_1 m_2 (1 - e^{x_1 - x_2})^2 + m_1 m_3 (1 - e^{x_1 - x_3})^2 + m_2 m_3 (1 - e^{x_2 - x_3})^2] z^2 \\ &\quad - [m_1 m_2 m_3 (1 - e^{x_1 - x_2})^2 (1 - e^{x_2 - x_3})^2] z^3. \end{aligned}$$

Remark 2.8. Although A_2 , for instance, has the same form for all n considered as a function of $\{x_1, x_2, m_1, m_2\}$, the coefficient $A_2(t; z)$ appearing in the rightmost interval $x_2 < x < \infty$ in the two-peakon wavefunction is not the same as the coefficient $A_2(t; z)$ appearing in the interval $x_2 < x < x_3$ in the three-peakon wavefunction, since the functions $\{x_k(t), m_k(t)\}$ are not the same in the two cases. In our presentation we always consider n to be fixed (but arbitrary), so hopefully no confusion will arise from this slight notational ambiguity.

The rightmost interval $x > x_n$ will be of particular interest, so we define

$$\begin{pmatrix} A(t; z) \\ B(t; z) \\ C(t; z) \end{pmatrix} = \begin{pmatrix} A_n(t; z) \\ B_n(t; z) \\ C_n(t; z) \end{pmatrix}. \quad (2.27)$$

2.6 Time dependence of ψ

Now we take a closer look at the time dependence of the coefficients $A_k(t; z)$, $B_k(t; z)$, $C_k(t; z)$ appearing in the peakon wavefunction ψ of Definition 2.6.

We know that $\psi(x, t; z)$ must satisfy equation (2.15b):

$$\psi_t = [z^{-1}(1 - \partial_x^2) + u_x - u\partial_x] \psi,$$

where $u(x, t) = \sum_1^n m_i(t) e^{-|x - x_i(t)|}$. We claimed before that our choice

$$(A_0(t; z), B_0(t; z), C_0(t; z)) = (1, 0, 0)$$

is consistent with (2.15b). This is easy to see; in the leftmost interval $x < x_1(t)$ we have $u(x, t) = \sum_1^n m_i(t) e^{x - x_i(t)}$ ($= u_x$) and $\psi(x, t; z) = e^x$ ($= \psi_x = \psi_{xx}$), hence both sides of (2.15b) vanish for $x < x_1(t)$.

Consider now the rightmost interval $x > x_n(t)$, where we have $u(x, t) = \sum_1^n m_i(t) e^{x_i(t)-x}$ ($= -u_x$) and $\psi = A(t; z) e^x + B(t; z) + C(t; z) e^{-x}$. Inserting this into (2.15b) yields

$$\begin{aligned} A_t e^x + B_t + C_t e^{-x} &= \left[\frac{1 - \partial_x^2}{z} - e^{-x} \left(\sum_1^n m_i e^{x_i} \right) (1 + \partial_x) \right] (A e^x + B + C e^{-x}) \\ &= 0 e^x + \left(\frac{B}{z} - 2A \sum_1^n m_i e^{x_i} \right) + \left(-B \sum_1^n m_i e^{x_i} \right) e^{-x}, \end{aligned}$$

which proves the following:

Proposition 2.9. *When $\{x_k, m_k\}_{k=1}^n$ evolve according to the DP peakon equations (2.5), the coefficients in the corresponding peakon wavefunction $\psi(x, t; z)$ evolve according to the equations*

$$A_t = 0, \quad B_t = \frac{B}{z} - 2A M_+, \quad C_t = -B M_+, \quad (2.28)$$

where $M_+ = \sum_{i=1}^n m_i e^{x_i}$.

In particular, the polynomial $A(z)$ is independent of t , so that its coefficients are constants of motion for (2.5). If we write

$$A(z) = 1 - M_1 z + M_2 z^2 - \cdots + (-1)^n M_n z^n, \quad (2.29)$$

then equation (2.26) gives the explicit formula (2.30) for M_p below. It is also useful to write $A(z) = \prod_{j=1}^n (1 - \frac{z}{\lambda_j})$, where $\{\lambda_k\}$ by definition are the zeros of $A(z)$ (see Section 2.7), from which we see that M_p is the p th elementary symmetric polynomial in $\{1/\lambda_k\}$.

Theorem 2.10. *The DP n -peakon equations (2.5) admit n functionally independent constants of motion M_1, \dots, M_n , given explicitly by*

$$M_p = \sum_{I \in \binom{[1, n]}{p}} \left(\prod_{i \in I} m_i \right) \left(\prod_{j=1}^{p-1} (1 - e^{x_{i_j} - x_{i_{j+1}}})^2 \right), \quad (2.30)$$

where $\binom{[1, n]}{p}$ is the set of all p -element subsets $I = \{i_1 < \cdots < i_p\}$ of $\{1, \dots, n\}$.

Example 2.11. The constants of motion in the case $n = 3$ are

$$\begin{aligned} M_1 &= m_1 + m_2 + m_3, \\ M_2 &= m_1 m_2 (1 - e^{x_1 - x_2})^2 + m_1 m_3 (1 - e^{x_1 - x_3})^2 + m_2 m_3 (1 - e^{x_2 - x_3})^2, \\ M_3 &= m_1 m_2 m_3 (1 - e^{x_1 - x_2})^2 (1 - e^{x_2 - x_3})^2. \end{aligned}$$

(Cf. the formula for A_3 in Example 2.7.)

2.7 The peakon spectral problem

In Definition 2.5 for the wavefunction ψ , we viewed (2.15a) (for fixed t and z) as an initial value problem, with initial data given by the asymptotic condition at $x = -\infty$. By imposing constraints both at $x = -\infty$ and at $x = +\infty$, we turn (2.15a) (for fixed t) into a spectral problem instead: given $m(x) \geq 0$, find the eigenvalues z such that

$$\begin{aligned} \psi_x(x) - \psi_{xx}(x) &= z m(x) \psi(x), \\ e^x \psi(x) &\rightarrow 0 \quad \text{as } x \rightarrow -\infty, \\ \psi_x(x) + \psi(x) &\rightarrow 0 \quad \text{as } x \rightarrow -\infty, \\ e^{-x} \psi(x) &\rightarrow 0 \quad \text{as } x \rightarrow +\infty \end{aligned} \tag{2.31}$$

has nontrivial solutions $\psi(x)$. These particular boundary conditions, which might seem slightly mysterious at this point, will turn out very natural in the setting of the cubic string; see Theorem 3.1. In any case, they take a much simpler form in the peakon case, which is what we are interested in here.

Theorem 2.12. *Let $m = 2 \sum_{i=1}^n m_i \delta_{x_i}$ be a discrete measure with $m_i > 0$ and $x_1 < \dots < x_n$. In this case, the spectral problem (2.31) has n distinct positive eigenvalues*

$$0 < \lambda_1 < \dots < \lambda_n.$$

Proof. We saw in the previous section that the solution to (2.15a) in the discrete case is given by (2.20) and (2.22), and then the boundary conditions in (2.31) reduce to

$$C_0 = 0, \quad B_0 = 0, \quad A_n = 0.$$

It is clear that we may normalize ψ by taking $A_0 = 1$, and then $A_n(z)$ is exactly the polynomial we considered before; its coefficients depend on the x_k 's and m_k 's, and are given explicitly by (2.26) with $k = n$. Consequently, the eigenvalues of (2.31) are in the discrete case simply the zeros of this n th degree polynomial $A(z) = A_n(z)$.

Now let $\{x_k, m_k\}$ evolve according to the peakon equations (2.5). Even though this changes the measure m , the eigenvalues of (2.31) remain unchanged, since the coefficients of $A(z)$ are constants of motion according to Section 2.6.

As $t \rightarrow \infty$, the peakons scatter as described by Theorem 2.4: the m_k 's tend to distinct positive values $m_k(\infty)$, and $x_{k+1} - x_k \rightarrow \infty$. By Theorem 2.10, this reduces the coefficients of $A(z)$ to the elementary symmetric polynomials in $\{m_k(\infty)\}$, so that $A(z) = \prod_{k=1}^n (1 - z m_k(\infty))$. It follows that the eigenvalues are $\{1/m_k(\infty)\}$, hence positive and distinct. \square

Remark 2.13. Since $m_1(\infty) < \dots < m_n(\infty)$ and $m_1(-\infty) > \dots > m_n(-\infty)$ by Theorem 2.4, and since the argument of the above proof holds equally well as $t \rightarrow -\infty$, it follows that $m_{n+1-k}(\infty) = 1/\lambda_k = m_k(-\infty)$. In other words, the same asymptotic velocities occur as $t \rightarrow +\infty$ and $t \rightarrow -\infty$, but in reverse order. We will study asymptotic peakon properties of this kind in more detail in Section 2.9 after giving the explicit formulas for the n -peakon solution.

It is clear from the isospectral deformation argument in the proof above that the numbers $\{x_k, m_k\}_{k=1}^n$ are not uniquely determined by the eigenvalues λ_k alone. The eigenvalues must be supplemented by further spectral data in order for the inverse spectral problem of reconstructing $m(x)$ to have a unique solution.

Definition 2.14 (Extended spectral data $\{\lambda_k, b_k, c_k\}$). Given a point in the peakon sector \mathcal{P} ,

$$x_1 < \cdots < x_n \quad \text{and} \quad m_1, \dots, m_n > 0,$$

we assign to it a $3n$ -tuple of numbers $\{\lambda_k, b_k, c_k\}_{k=1}^n$ via the following construction. Let the polynomials $(A(z), B(z), C(z)) = (A_n(z), B_n(z), C_n(z))$ be given by (2.26) with $k = n$. Let

$$0 < \lambda_1 < \cdots < \lambda_n \tag{2.32}$$

be the zeros of $A(z)$ (which are positive and distinct by Theorem 2.12), and let b_k and c_k be the residues in the partial fraction decompositions

$$\frac{-B(z)}{2z A(z)} = \sum_{k=1}^n \frac{b_k}{z - \lambda_k}, \quad \frac{C(z) - B(z)}{2z A(z)} = \sum_{k=1}^n \frac{c_k}{z - \lambda_k}. \tag{2.33}$$

For future convenience (see Theorem 3.5), also let

$$\lambda_0 = 0, \quad b_0 = 1, \quad c_0 = \frac{1}{2}. \tag{2.34}$$

Theorem 2.15. *When $\{x_k, m_k\}_{k=1}^n$ evolve according to the DP peakon equations (2.5), the extended spectral data $\{\lambda_k, b_k, c_k\}_{k=1}^n$ evolve according to the equations*

$$\dot{\lambda}_k = 0, \quad \dot{b}_k = \frac{b_k}{\lambda_k}, \quad \dot{c}_k = \dot{b}_k + \sum_{j=1}^n \frac{b_k b_j}{\lambda_j}. \tag{2.35}$$

Proof. We have already seen that the eigenvalues λ_k are constant. Let

$$\omega(z) = -\frac{B(z)}{2z A(z)} = \sum_{k=1}^n \frac{b_k}{z - \lambda_k}$$

and

$$\zeta(z) = \frac{C(z)}{2z A(z)} = \sum_{k=1}^n \frac{c_k - b_k}{z - \lambda_k}.$$

Equation (2.26) shows that $A(0) = 1$ and $B(z) = 2M_+ z + O(z^2)$, so that $\omega(0) = -M_+$, where $M_+ = \sum_{k=1}^n m_k e^{x_k}$. The time evolution of $A(z)$, $B(z)$, $C(z)$, given by (2.28), now translates into

$$\omega_t(z) = -\frac{B_t(z)}{2z A(z)} = -\frac{B(z)/z - 2A(z)M_+}{2z A(z)} = \frac{\omega(z) - \omega(0)}{z}$$

and

$$\zeta_t(z) = \frac{C_t(z)}{2z A(z)} = \frac{-B(z)M_+}{2z A(z)} = -\omega(z)\omega(0).$$

Comparing residues at $z = \lambda_k$ on the right- and left-hand sides of these equations one obtains at once (2.35). \square

Corollary 2.16. *The c_k 's are determined by the λ_k 's and the b_k 's through the formula*

$$c_k = \lambda_k b_k \sum_{j=0}^n \frac{b_j}{\lambda_j + \lambda_k} \quad (k = 1, \dots, n). \quad (2.36)$$

(Since $c_0 = 1/2$ and $b_0 = 1$ by definition, the formula holds trivially also in the case $k = 0$, if interpreted as being a limit as $\lambda_0 \rightarrow 0+$.)

Proof. From the second equation in (2.35) it is immediate that $b_k(t) = b_k(0)e^{t/\lambda_k}$. Inserting this into the third equation in (2.35) we get

$$\frac{d}{dt}(c_k(t) - b_k(t)) = \sum_{j=1}^n \frac{b_k(0)b_j(0)}{\lambda_j} \exp\left(\frac{t}{\lambda_k} + \frac{t}{\lambda_j}\right).$$

From (2.26) and Theorem 2.4 it is clear that $C(z) \rightarrow 0$ as $t \rightarrow -\infty$, hence $\zeta(z)$ and its residues $c_k - b_k$ also vanish as $t \rightarrow -\infty$. Thus, integrating the ODE above from $-\infty$ to t yields

$$c_k(t) - b_k(t) = \sum_{j=1}^n \frac{b_k(0)b_j(0)}{\lambda_j \left(\frac{1}{\lambda_k} + \frac{1}{\lambda_j}\right)} \exp\left(\frac{t}{\lambda_k} + \frac{t}{\lambda_j}\right) = \sum_{j=1}^n \frac{\lambda_k b_k(t)b_j(t)}{\lambda_j + \lambda_k}.$$

Using the definitions $\lambda_0 = 0$ and $b_0 = 1$, the b_k on the left-hand side can be incorporated in the sum by summing from $j = 0$ instead of $j = 1$, and this gives (2.36). \square

Theorem 2.17. *The b_k 's and c_k 's are positive.*

Proof. First we show that $b_k \neq 0$. By (2.33), $b_k = -B(\lambda_k)/2\lambda_k A'(\lambda_k)$, so it is enough to show that $B(\lambda_k) \neq 0$. Let $\psi_k(x)$ denote the eigenfunction corresponding to the eigenvalue λ_k . Since m has compact support, ψ_k and all its derivatives vanish as $x \rightarrow -\infty$, while as $x \rightarrow +\infty$ the derivatives vanish and $\psi_k(\infty) = B(\lambda_k)$. Thus it suffices to show that $\psi_k(\infty) \neq 0$. Since ψ_k satisfies $\psi_x - \psi_{xxx} = \lambda_k m \psi$ we readily obtain, after the standard steps of multiplying by ψ , integrating and using the boundary conditions, that $\frac{1}{2}\psi_k(\infty)^2 = \lambda_k \int_{-\infty}^{\infty} m \psi_k^2 dx > 0$. This proves that $b_k \neq 0$ for all k .

To prove $b_k > 0$ we use induction on the number of peakons n . The case $n = 1$ follows from explicit formulas. We assume the claim to be valid for an arbitrary configuration of $n - 1$ peakons. Then we use the following deformation argument.

From the explicit formula (2.26) for $A(z)$ and $B(z)$, we see that by letting x_1 tend to $-\infty$ (keeping the other variables fixed) we can make the coefficients in the rational function

$$\omega(z) = -\frac{B(z)}{2z A(z)} = \sum_{k=1}^n \frac{b_k}{z - \lambda_k}$$

come arbitrarily close to the coefficients in the function

$$\tilde{\omega} = \frac{-(1 - zm_1)\tilde{B}(z)}{2z(1 - zm_1)\tilde{A}(z)},$$

(where \tilde{B} and \tilde{A} are the polynomials computed for the $(n-1)$ -peakon configuration consisting of all but the first peakon), which has a partial fraction decomposition

$$\frac{0}{z - \kappa_1} + \sum_{k=2}^n \frac{d_k}{z - \kappa_k} \quad (\kappa_1 := 1/m_1).$$

(If $1/m_1$ happens to coincide with one of the eigenvalues $\kappa_2, \dots, \kappa_n$ of the $(n-1)$ -peakon configuration, then just change m_1 a little before x_1 is sent to $-\infty$ so that all κ_k 's are distinct.) The poles and residues $\{\lambda_k, b_k\}_1^n$ of the rational function $\omega(z)$ depend continuously on its coefficients, which in turn depend continuously on $\{x_k, m_k\}_1^n$. Since the b_k 's are nonzero, it follows that they cannot change sign during this deformation, which brings the λ_k 's arbitrarily close to the κ_k 's, and b_1, \dots, b_n arbitrarily close to $0, d_2, \dots, d_n$. By the induction hypothesis d_k 's are all positive. This implies that b_2, \dots, b_n must have been positive to begin with.

Moreover, we know from the proof of Theorem 2.15 that the strictly positive function $M_+ = \sum_{k=1}^n m_k e^{x_k}$ satisfies $M_+ = -\omega(0) = \sum_{k=1}^n b_k / \lambda_k$. Letting the peakons evolve according to the DP equation, $b_1(t) = b_1(0)e^{t/\lambda_1}$ is the dominant term in this sum as $t \rightarrow \infty$. This leads to a contradiction if $b_1 < 0$, hence $b_1 > 0$.

Finally, since all λ_k 's and b_k 's are positive, equation (2.36) shows that all c_k 's are positive as well. \square

2.8 The explicit n -peakon solution

In the extended spectral data $\{\lambda_k, b_k, c_k\}_{k=1}^n$ investigated in the previous section, the quantities c_k play only an auxiliary role (although important), since they are determined by the λ_k 's and b_k 's. The primary objects are $\{\lambda_k, b_k\}_{k=1}^n$, which we will refer to as *spectral variables*. Definition 2.14 gives the prescription for going from peakon variables $\{x_k, m_k\}$ to spectral variables $\{\lambda_k, b_k\}$. We can view this as a map

$$\begin{aligned} \mathcal{S} : \mathcal{P} &\rightarrow \mathcal{R} \\ (x, m) &\mapsto (\lambda, b), \end{aligned} \tag{2.37}$$

where the “pure peakon sector” \mathcal{P} is defined by (2.8), while the region of allowable spectral data is

$$\mathcal{R} = \{(\lambda, b) \in \mathbf{R}^{2n} : 0 < \lambda_1 < \dots < \lambda_n, \text{ all } b_i > 0\}. \tag{2.38}$$

Theorems 2.12 and 2.17 show that \mathcal{S} does map \mathcal{P} into \mathcal{R} . To justify the terminology “spectral variables”, we need to show that \mathcal{S} is injective; in fact we will show that \mathcal{S} is a bijection of \mathcal{P} onto \mathcal{R} , and find \mathcal{S}^{-1} explicitly. We will do this by transforming the spectral problem (2.31) to the equivalent cubic string spectral problem (3.3) and studying the inverse spectral problem in that setting. We will state the main result in Theorem 2.22 in order to explore some of its consequences below, but the central part of its proof will have to wait until Section 4.4.

Once we know how to invert the change of variables (2.37), we can also solve the DP n -peakon equations (2.5), since in new variables they simply take the form of the linear equations (2.35). Using an evolution operator describing the solution to (2.35),

$$\begin{aligned}\Phi_t : \mathcal{R} &\rightarrow \mathcal{R} \\ (\lambda_k, b_k) &\mapsto (\lambda_k, b_k e^{t/\lambda_k}),\end{aligned}\tag{2.39}$$

we can write the n -peakon solution as

$$(x(t), m(t)) = \mathcal{S}^{-1} \circ \Phi_t \circ \mathcal{S}(x(0), m(0)).\tag{2.40}$$

However, finding the formulas for \mathcal{S}^{-1} is quite a difficult problem, which will occupy us for most of the remainder of the paper. We start by introducing necessary notation.

Definition 2.18 (Various notation). For $k \geq 2$, let

$$\Delta(x_1, \dots, x_k) = \prod_{i < j} (x_i - x_j), \quad \Gamma(x_1, \dots, x_k) = \prod_{i < j} (x_i + x_j).\tag{2.41}$$

For $k \geq 0$, we recall that $\binom{[1, n]}{k}$ denotes the set of k -element subsets $I = \{i_1 < \dots < i_k\}$ of the integer interval $[1, n] = \{1, \dots, n\}$. For $I \in \binom{[1, n]}{k}$, let

$$\Delta_I = \Delta(\lambda_{i_1}, \dots, \lambda_{i_k}), \quad \Gamma_I = \Gamma(\lambda_{i_1}, \dots, \lambda_{i_k}),\tag{2.42}$$

with the special cases $\Delta_\emptyset = \Gamma_\emptyset = \Delta_{\{i\}} = \Gamma_{\{i\}} = 1$. For two disjoint ordered sets I and J we will also use occasionally

$$\Delta_{I, J}^2 = \prod_{i \in I, j \in J} (\lambda_i - \lambda_j)^2, \quad \Gamma_{I, J} = \prod_{i \in I, j \in J} (\lambda_i + \lambda_j),\tag{2.43}$$

with the convention that the symbols equal 1 if one or both sets are empty. (Note that $\Gamma_{I \cup J} = \Gamma_I \Gamma_J \Gamma_{I, J}$, and similarly for Δ^2 .) Furthermore, let

$$\lambda_I = \prod_{i \in I} \lambda_i, \quad b_I = \prod_{i \in I} b_i,$$

with the proviso $\lambda_\emptyset = b_\emptyset = 1$.

Using this notation we now define the symmetric functions that appear in the explicit solution formulas for DP peakons.

Definition 2.19 (Symmetric functions).

$$U_k = \sum_{I \in \binom{[1,n]}{k}} \frac{\Delta_I^2}{\Gamma_I} b_I, \quad V_k = \sum_{I \in \binom{[1,n]}{k}} \frac{\Delta_I^2}{\Gamma_I} \lambda_I b_I, \quad (2.44)$$

and

$$W_k = \begin{vmatrix} U_k & V_{k-1} \\ U_{k+1} & V_k \end{vmatrix} = U_k V_k - U_{k+1} V_{k-1}. \quad (2.45)$$

(To be explicit, $U_0 = V_0 = 1$, and $U_k = V_k = 0$ for $k < 0$ or $k > n$.)

We can evaluate W_k explicitly in terms of (λ, b) as follows.

Lemma 2.20.

$$\begin{aligned} W_k = & \sum_{I \in \binom{[1,n]}{k}} \frac{\Delta_I^4}{\Gamma_I^2} \lambda_I b_I^2 \\ & + \sum_{m=1}^k \sum_{\substack{I \in \binom{[1,n]}{k-m} \\ J \in \binom{[1,n]}{2m} \\ I \cap J = \emptyset}} b_I^2 b_J \left\{ 2^{m+1} \frac{\Delta_I^4 \Delta_{I,J}^2 \lambda_{I \cup J}}{\Gamma_I \Gamma_{I \cup J}} \right. \\ & \left. \times \left(\sum_{\substack{C \cup D = J \\ |C|=|D|=m \\ \min(C) < \min(D)}} \Delta_C^2 \Delta_D^2 \Gamma_C \Gamma_D \right) \right\}. \end{aligned} \quad (2.46)$$

Proof. For brevity, let $\Psi_I = \Delta_I^2 / \Gamma_I$ and $\Psi_{I,J} = \Delta_{I,J}^2 / \Gamma_{I,J}$. Consider first the contribution from $U_k V_k$, which is a sum of terms $(\Psi_A b_A)(\Psi_B \lambda_B b_B)$, with $A, B \in \binom{[1,n]}{k}$. Since b_A and b_B each contain only distinct b_i 's, we have $b_A b_B = b_I^2 b_J$ where $I = A \cap B$ and $J = A \triangle B$ (symmetric difference); note that $I \cap J = \emptyset$. The terms for which $I = A = B$ and $J = \emptyset$ give rise to the first sum in (2.46). Otherwise, $|I| = k - m$ and $|J| = 2m$ for some $1 \leq m \leq k$, and the sets $E = A \setminus I$ and $F = B \setminus I$ partition $J = E \cup F$ into two disjoint sets with $|E| = |F| = m$. With this notation we have $(\Psi_A b_A)(\Psi_B \lambda_B b_B) = (\Psi_I^2 \Psi_{I,J} \lambda_I)(\Psi_E \Psi_F \lambda_F)(b_I^2 b_J)$.

The contribution from $U_{k+1} V_{k-1}$ is similar, except that $A \in \binom{[1,n]}{k+1}$ and $B \in \binom{[1,n]}{k-1}$ so that the case $A = B$ can never happen. In the disjoint partition $J = G \cup H$, where $G = A \setminus I$ and $H = B \setminus I$, we have $|G| = m + 1$ and $|H| = m - 1$. Hence,

$$\begin{aligned} W_k = & \sum_{I \in \binom{[1,n]}{k}} \Psi_I^2 \lambda_I b_I^2 \\ & + \sum_{m=1}^k \sum_{\substack{I \in \binom{[1,n]}{k-m} \\ J \in \binom{[1,n]}{2m} \\ I \cap J = \emptyset}} b_I^2 b_J \Psi_I^2 \Psi_{I,J} \lambda_I \left(\sum_{\substack{E \cup F = J \\ |E|=|F|=m}} \Psi_E \Psi_F \lambda_F - \sum_{\substack{G \cup H = J \\ |G|=m+1 \\ |H|=m-1}} \Psi_G \Psi_H \lambda_H \right). \end{aligned}$$

The expression in brackets equals $1/\Gamma_J$ times the left-hand side of the symmetric polynomial identity

$$\begin{aligned} \sum_{\substack{E \cup F = J \\ |E| = |F| = m}} \Delta_E^2 \Delta_F^2 \Gamma_{E,F} \lambda_F - \sum_{\substack{G \cup H = J \\ |G| = m+1 \\ |H| = m-1}} \Delta_G^2 \Delta_H^2 \Gamma_{G,H} \lambda_H \\ = 2^{m+1} \lambda_J \left(\sum_{\substack{C \cup D = J \\ |C| = |D| = m \\ \min(C) < \min(D)}} (\Delta_C \Delta_D)^2 \Gamma_C \Gamma_D \right), \end{aligned} \quad (2.47)$$

substitution of which yields the desired formula (2.46).

To prove (2.47) we use induction on $m = |J|/2$. Without loss of generality, we can take $J = \{1, 2, \dots, 2m\}$. When $m = 1$, both sides reduce to $4\lambda_1\lambda_2$. For $m > 1$, evaluating (2.47) at $\lambda_{2m-1} = \lambda_{2m} = c$ yields $4c^2(\prod_{j=1}^{2m-2}(\lambda_j - c)^2(\lambda_j + c))$ times the left- and right-hand sides, respectively, of the $m-1$ case of (2.47), which holds by the induction hypothesis. Hence, by symmetry, (2.47) holds whenever two variables are equal. This means that the difference between the left- and the right-hand sides is divisible by Δ_J , but since that difference is a symmetric polynomial it must be divisible by Δ_J^2 . However, the difference is of degree $3m-1$ in λ_1 , while Δ_J^2 has degree $2(2m-1)$ in λ_1 . Since $3m-1 < 4m-2$ if $m > 1$, we conclude that the difference is identically zero, and (2.47) is proved. \square

Remark 2.21. Similar polynomial identities will be used later in the proof of Lemma 4.10; see also Appendix B. The constraint $\min(C) < \min(D)$ eliminates redundancy; if it is removed, then every term appears twice in the sum, and 2^{m+1} should be changed to 2^m .

Theorem 2.22 (Inverse of \mathcal{S}). *The change of variables $\mathcal{S} : \mathcal{P} \rightarrow \mathcal{R}$ is a bijection. Its inverse*

$$\begin{aligned} \mathcal{S}^{-1} : \mathcal{R} &\rightarrow \mathcal{P} \\ (\lambda, b) &\mapsto (x, m) \end{aligned} \quad (2.48)$$

is given by

$$x_{k'} = \log \frac{U_k}{V_{k-1}}, \quad m_{k'} = \frac{(U_k)^2 (V_{k-1})^2}{W_k W_{k-1}} \quad (k = 1, \dots, n), \quad (2.49)$$

where $k' = n + 1 - k$.

Proof. For the proof, denote the map (2.49) by \mathcal{T} instead of \mathcal{S}^{-1} . In Section 4.4 we will prove that $\mathcal{T} \circ \mathcal{S} = \text{id}_{\mathcal{P}}$, using the solution of the inverse spectral problem for the discrete cubic string. In words: provided that $(\lambda, b) = \mathcal{S}(x, m)$ is in the range of \mathcal{S} , we know that $(x, m) = \mathcal{T}(\lambda, b)$. To complete the proof, we need to show that the range of \mathcal{S} is all of \mathcal{R} (that is, all allowable spectral data really correspond to a peakon configuration).

First note that \mathcal{T} maps \mathcal{R} into the pure peakon sector \mathcal{P} . Indeed, from the definitions of U_k and V_k , and from the explicit formula (2.46) for W_k , it is clear that all U_k , V_k , and W_k (for $1 \leq k \leq n$) are positive when $(\lambda, b) \in \mathcal{R}$; then (2.49) produces a point $(x, m) = \mathcal{T}(\lambda, b)$ with all $m_i > 0$ (obviously), and also with $x_1 < \dots < x_n$ since

$$W_k > 0 \iff \frac{U_{k+1}}{V_k} < \frac{U_k}{V_{k-1}} \iff x_{k'-1} < x_{k'}.$$

Recall the definition of \mathcal{S} : given $(x, m) \in \mathcal{P}$ we form the polynomials $A(z)$ and $B(z)$, whose coefficients are Laurent polynomials in $\{e^{x_k}, m_k\}_{k=1}^n$. Write $A(z; (x, m))$ and $B(z; (x, m))$ to denote this dependence explicitly. The λ_k 's are defined as the zeros of $A(z)$, and the b_k 's as the residues in $-B(z)/2zA(z)$.

Now we use the algebraic nature of the mappings involved. The coefficients in the polynomial $A(z; \mathcal{T}(\lambda, b))$ are rational functions of $\{\lambda_k, b_k\}_{k=1}^n$ which, since $\mathcal{T} \circ \mathcal{S} = \text{id}_{\mathcal{P}}$, agree with the coefficients in $\prod_{k=1}^n (1 - z/\lambda_k)$ on the range of \mathcal{S} (which is an open set in \mathcal{R} since \mathcal{S} is a homeomorphism). Hence $A(z; \mathcal{T}(\lambda, b)) = \prod_{k=1}^n (1 - z/\lambda_k)$ identically as a rational function of (λ, b) ; in particular this relation holds for all points $(\lambda, b) \in \mathcal{R}$. Similarly, the coefficients of $B(z; \mathcal{T}(\lambda, b))$ agree with $-2z(\sum_{i=1}^n b_i/(z - \lambda_i)) \prod_{k=1}^n (1 - z/\lambda_k)$ on the range of \mathcal{S} , hence on all of \mathcal{R} since they are rational functions. This proves that $\mathcal{S} \circ \mathcal{T} = \text{id}_{\mathcal{R}}$. Consequently, the range of \mathcal{S} is \mathcal{R} , and $\mathcal{T} = \mathcal{S}^{-1}$, as claimed. \square

Corollary 2.23 (The n -peakon solution). *The solution $\{x_k(t), m_k(t)\}_{k=1}^n$ of the Degasperis–Procesi n -peakon equations (2.5) is given by (2.49) with*

$$b_i(t) = b_i(0)e^{t/\lambda_i} \quad (i = 1, \dots, n),$$

where the constants $\{\lambda_i, b_i(0)\}_{i=1}^n$ are determined from the initial conditions $\{x_k(0), m_k(0)\}_{k=1}^n$.

Example 2.24 (The three-peakon solution). For $n = 1$ and $n = 2$, equation (2.49) reduces to (2.6) and (2.7), respectively. The solution for $n = 3$, when simplified by taking into account that some of the U_k 's and V_k 's equal 0 or 1, takes the form

$$\begin{aligned} x_3(t) &= \log U_1, & x_2(t) &= \log \frac{U_2}{V_1}, & x_1(t) &= \log \frac{U_3}{V_2}, \\ m_3(t) &= \frac{(U_1)^2}{W_1}, & m_2(t) &= \frac{(U_2)^2(V_1)^2}{W_2W_1}, & m_1(t) &= \frac{U_3(V_2)^2}{V_3W_2}, \end{aligned} \quad (2.50)$$

where, with $b_i = b_i(0)e^{t/\lambda_i}$,

$$\begin{aligned}
U_1 &= b_1 + b_2 + b_3, \\
V_1 &= \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3, \\
U_2 &= \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1 + \lambda_2} b_1 b_2 + \frac{(\lambda_1 - \lambda_3)^2}{\lambda_1 + \lambda_3} b_1 b_3 + \frac{(\lambda_2 - \lambda_3)^2}{\lambda_2 + \lambda_3} b_2 b_3, \\
V_2 &= \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1 + \lambda_2} \lambda_1 \lambda_2 b_1 b_2 + \frac{(\lambda_1 - \lambda_3)^2}{\lambda_1 + \lambda_3} \lambda_1 \lambda_3 b_1 b_3 + \frac{(\lambda_2 - \lambda_3)^2}{\lambda_2 + \lambda_3} \lambda_2 \lambda_3 b_2 b_3, \\
U_3 &= \frac{(\lambda_1 - \lambda_2)^2 (\lambda_1 - \lambda_3)^2 (\lambda_2 - \lambda_3)^2}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)} b_1 b_2 b_3, \\
V_3 &= \frac{(\lambda_1 - \lambda_2)^2 (\lambda_1 - \lambda_3)^2 (\lambda_2 - \lambda_3)^2}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)} \lambda_1 \lambda_2 \lambda_3 b_1 b_2 b_3,
\end{aligned}$$

and consequently

$$\begin{aligned}
W_1 &= U_1 V_1 - U_2 \\
&= \lambda_1 b_1^2 + \lambda_2 b_2^2 + \lambda_3 b_3^2 + \frac{4\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} b_1 b_2 + \frac{4\lambda_1 \lambda_3}{\lambda_1 + \lambda_3} b_1 b_3 + \frac{4\lambda_2 \lambda_3}{\lambda_2 + \lambda_3} b_2 b_3, \\
W_2 &= U_2 V_2 - U_3 V_1 \\
&= \frac{(\lambda_1 - \lambda_2)^4}{(\lambda_1 + \lambda_2)^2} \lambda_1 \lambda_2 (b_1 b_2)^2 + \frac{(\lambda_1 - \lambda_3)^4}{(\lambda_1 + \lambda_3)^2} \lambda_1 \lambda_3 (b_1 b_3)^2 + \frac{(\lambda_2 - \lambda_3)^4}{(\lambda_2 + \lambda_3)^2} \lambda_2 \lambda_3 (b_2 b_3)^2 \\
&\quad + \frac{4\lambda_1 \lambda_2 \lambda_3 b_1 b_2 b_3}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)} \times \left((\lambda_1 - \lambda_2)^2 (\lambda_1 - \lambda_3)^2 b_1 \right. \\
&\quad \left. + (\lambda_2 - \lambda_1)^2 (\lambda_2 - \lambda_3)^2 b_2 + (\lambda_3 - \lambda_1)^2 (\lambda_3 - \lambda_2)^2 b_3 \right).
\end{aligned}$$

We take the opportunity here to correct a mistake in our previous paper [15], where the denominator W_1 in $m_3(t)$ was incorrectly stated with $(\lambda_i + \lambda_j)^2$ instead of $\lambda_i + \lambda_j$.

2.9 Peakon asymptotics and scattering

We will now extract information from the explicit solution formulas (2.49) about how the peakons interact, and about their asymptotic behavior as $t \rightarrow \pm\infty$. Recall that we number the eigenvalues so that $0 < \lambda_1 < \dots < \lambda_n$, and consequently

$$0 < \frac{1}{\lambda_n} < \dots < \frac{1}{\lambda_1}.$$

Also recall we use $k' = n + 1 - k$ to denote the complementary index to k .

Theorem 2.25 (Asymptotics).

$$\begin{aligned} x_k(t) &\sim \frac{t}{\lambda_k} + \log b_k(0) + \sum_{i=k+1}^n \log \frac{(\lambda_i - \lambda_k)^2}{(\lambda_i + \lambda_k)\lambda_i}, \quad \text{as } t \rightarrow -\infty, \\ x_{k'}(t) &\sim \frac{t}{\lambda_k} + \log b_k(0) + \sum_{i=1}^{k-1} \log \frac{(\lambda_i - \lambda_k)^2}{(\lambda_i + \lambda_k)\lambda_i}, \quad \text{as } t \rightarrow +\infty, \end{aligned} \quad (2.51)$$

and

$$\begin{aligned} m_k(t) &= 1/\lambda_k + O(e^{t\delta_k}), \quad \text{as } t \rightarrow -\infty, \\ m_{k'}(t) &= 1/\lambda_k + O(e^{-t\delta_k}), \quad \text{as } t \rightarrow +\infty, \end{aligned} \quad (2.52)$$

where

$$\delta_k = \begin{cases} \frac{1}{\lambda_1} - \frac{1}{\lambda_2}, & k = 1, \\ \min\left(\frac{1}{\lambda_{k-1}} - \frac{1}{\lambda_k}, \frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}}\right), & k = 2, \dots, n-1, \\ \frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n}, & k = n. \end{cases} \quad (2.53)$$

Proof. This is simply a matter of identifying the dominant terms among the b_k 's. Because of the ordering of the eigenvalues, $b_1(t) = b_1(0)e^{t/\lambda_1}$ dominates as $t \rightarrow +\infty$ and $b_n(t) = b_n(0)e^{t/\lambda_n}$ dominates as $t \rightarrow -\infty$. More generally, $b_1 b_2 \cdots b_k$ grows exponentially faster as $t \rightarrow +\infty$ than any other product $b_{i_1} b_{i_2} \cdots b_{i_k}$ with $i_1 < \cdots < i_k$. It follows that, as $t \rightarrow +\infty$,

$$U_k \sim \frac{(\Delta_{12\dots k})^2}{\Gamma_{12\dots k}} b_1 b_2 \cdots b_k,$$

and

$$V_k \sim \frac{(\Delta_{12\dots k})^2}{\Gamma_{12\dots k}} \lambda_1 \lambda_2 \cdots \lambda_k b_1 b_2 \cdots b_k,$$

while (by similar reasoning) for $t \rightarrow -\infty$ each index on the right-hand side is replaced by its complementary value $i' = n + 1 - i$. Hence, as $t \rightarrow +\infty$,

$$\frac{U_k}{V_{k-1}} \sim \left(\frac{\Delta_{12\dots k}}{\Delta_{12\dots(k-1)}} \right)^2 \frac{\Gamma_{12\dots(k-1)}}{\Gamma_{12\dots k}} \frac{b_k}{\lambda_1 \lambda_2 \cdots \lambda_{k-1}} = b_k(0) e^{t/\lambda_k} \prod_{i=1}^{k-1} \frac{(\Delta_{ik})^2}{\Gamma_{ik} \lambda_i}.$$

Since $U_k/V_{k-1} = \exp(x_{k'})$, the case $t \rightarrow +\infty$ of (2.51) follows by taking logarithms. The case $t \rightarrow -\infty$ is similar.

As for (2.52), we have already seen (Remark 2.13) that $m_k(-\infty) = 1/\lambda_k = m_{k'}(+\infty)$, and this can of course also be concluded by comparing coefficients of the dominant term $(b_1 \cdots b_{k-1})^4 (b_k)^2$ (as $t \rightarrow +\infty$) in the numerator and the denominator of the formula for $m_{k'}(t)$. As for the correction term, for $1 < k < n$ a routine computation gives

$$m_{k'}(t) = \frac{1}{\lambda_k} \frac{(1 + A_k \frac{b_{k+1}}{b_k})^2 (1 + B_k \frac{b_k}{b_{k-1}})^2}{(1 + C_k \frac{b_{k+1}}{b_k})(1 + D_k \frac{b_k}{b_{k-1}})} + \text{lower-order terms}$$

as $t \rightarrow +\infty$, where A_k, B_k, C_k, D_k are some t -independent and computable coefficients. This implies

$$m_{k'}(t) - \frac{1}{\lambda_k} = \alpha_k \frac{b_{k+1}}{b_k} + \beta_k \frac{b_k}{b_{k-1}} + \text{lower-order terms}$$

which gives the stated result as $t \rightarrow +\infty$. A similar analysis applies to the cases $k = 1$ and $k = n$, where there are no terms involving b_{k-1} or b_{k+1} , respectively. Again, the case $t \rightarrow -\infty$ is similar. \square

Corollary 2.26 (Phase shifts). *The peakon with asymptotic velocity $1/\lambda_k$ as $t \rightarrow \pm\infty$ experiences the phase shift*

$$\begin{aligned} \lim_{t \rightarrow \infty} \left(x_{k'}(t) - \frac{t}{\lambda_k} \right) - \lim_{t \rightarrow -\infty} \left(x_k(t) - \frac{t}{\lambda_k} \right) = \\ = \sum_{i=1}^{k-1} \log \frac{(\lambda_i - \lambda_k)^2}{(\lambda_i + \lambda_k)\lambda_i} - \sum_{i=k+1}^n \log \frac{(\lambda_i - \lambda_k)^2}{(\lambda_i + \lambda_k)\lambda_i}. \end{aligned} \quad (2.54)$$

Remark 2.27. For every t there is a well-defined notion of what “the peakon at site k ” means, namely, the peaked wave $m_k(t) \exp(-|x - x_k(t)|)$ described by m_k and x_k . For other types of soliton equations, individual solitons are usually only identifiable as they scatter when $t \rightarrow \pm\infty$. In that case, the highest/fastest soliton, which is to the far left as $t \rightarrow -\infty$, reemerges to the far right as $t \rightarrow +\infty$ after some interactions during which the identities of the solitons are blurred, and similarly for the other solitons. According to this point of view, the “ k th fastest/highest peakon” (the one with asymptotic velocity $1/\lambda_k$) moves from occupying site k as $t \rightarrow -\infty$ to occupying site k' as $t \rightarrow +\infty$, and does not really have a well-defined identity during the interactions causing this change. It is the phase shift of this k th fastest peakon that is given in (2.54).

Remark 2.28. Note that the total phase shift in (2.54) equals the sum of phase shifts resulting from pairwise interactions; since the peakons reverse their order (according to the interpretation above), the k th fastest peakon is overtaken by the ones initially to its left (giving the contribution $\sum_{i=1}^{k-1}$) and overtakes the ones initially to its right (which contributes $-\sum_{i=k+1}^n$).

We see that the interaction of a faster peakon overtaking a slower one (with asymptotic speeds $c_f = 1/\lambda_f$ and $c_s = 1/\lambda_s$, respectively, where $f < s$ so that $c_f > c_s$) results in the slower peakon shifting by the amount

$$\log \frac{(\lambda_f - \lambda_s)^2}{(\lambda_f + \lambda_s)\lambda_f} = \log \frac{(c_f - c_s)^2}{(c_f + c_s)c_s},$$

which is positive iff $c_f > 3c_s$, while the shift of the faster peakon is

$$-\log \frac{(\lambda_s - \lambda_f)^2}{(\lambda_s + \lambda_f)\lambda_s} = \log \frac{(c_f + c_s)c_f}{(c_f - c_s)^2},$$

which is always positive. This generalizes the results obtained in [5] for the case $n = 2$.

Remark 2.29. The phase shifts obtained in the Camassa–Holm peakon case in [2], as well as finite Toda phase shifts obtained by Moser in [16], have a slightly different structure; namely, in the case of CH peakons we have (after some rescaling)

$$\sum_{i=1}^{k-1} \log \frac{(\lambda_i - \lambda_k)^2}{\lambda_i^2} - \sum_{i=k+1}^n \log \frac{(\lambda_i - \lambda_k)^2}{\lambda_i^2}.$$

2.10 A few comments about antipeakons

So far we have only been discussing the pure peakon case where all $m_i > 0$. Everything we have said applies (*mutatis mutandis*) also to the pure antipeakon case where all $m_i < 0$, because of the following natural symmetry of the problem.

Proposition 2.30 (Antipeakons). *Let $\{x_k(t), m_k(t)\}$ be a global solution to (2.5) satisfying Assumption 2.2 (pure peakons and ordering). Then*

$$m_k^*(t) = -m_{n+1-k}(-t), \quad x_k^*(t) = x_{n+1-k}(-t) \quad (2.55)$$

is a global solution to (2.5) satisfying $m_k^(t) < 0$, $x_1^*(t) > x_2^*(t) > \dots > x_n^*(t)$.*

Proof. Straightforward. \square

However, when both peakons and antipeakons are present, the situation is more complicated, because of the singularities that occur after finite time when a peakon travelling to the right collides with an antipeakon travelling to the left. Most of our proofs break down, since they rely on the existence of a global solution (Proposition 2.3) and its scattering properties as $t \rightarrow \pm\infty$ (Theorem 2.4).

We suspect that the spectrum is still real and simple in the presence of antipeakons, with the same number of positive (negative) eigenvalues as peakons (antipeakons). This is easy to prove for $n = 1$ and 2 , and also agrees with the analogous situation for the CH peakons [2]. However, in contrast to the CH case, the spectral measure $\mu = \sum_1^n b_k \delta_{\lambda_k}$ need *not* be positive in the mixed peakon-antipeakon case (it is not hard to find counterexamples for $n = 2$).

The explicit solution formulas (2.49) remain valid as long as they make sense, because of their algebraic nature. However, they break down completely if some $\lambda_i + \lambda_j = 0$; for example, in the case of a totally symmetric peakon-antipeakon collision (equation (6.13) in [5]). And even if all $\lambda_i + \lambda_j \neq 0$, the solution provided by (2.49) will in general not satisfy the ordering assumption $x_1 < \dots < x_n$, or even be defined, for all t . This is because U_k , V_k , and W_k need not remain positive (even if positive at $t = 0$) when some $b_k < 0$. In intervals $t \in (t_1, t_2)$ where the ordering assumption is violated, (2.49) gives a solution of (2.10) (now with some $d_{ij} = x_i - x_j < 0$), but this will not be a solution of the original peakon equations (2.5) containing $|d_{ij}|$ instead of d_{ij} .

All this is also in contrast to the CH case, where the solution formulas for $x_k(t)$ and $x_{k+1}(t)$ always automatically satisfy $x_k \leq x_{k+1}$ for all t , with equality

only at the instant of a peakon-antipeakon collision where $m_k(t)$ and $m_{k+1}(t)$ diverge to $\pm\infty$. This provides a continuation of the solution past the collision. It seems as if continuing DP peakon-antipeakon solutions past collisions would require some kind of gluing of different solutions from different time intervals, but the details are not clear at present.

3 The cubic string

The peakon spectral problem (2.31) is equivalent under a change of variables to what we have called the *cubic string* problem, an interesting non-selfadjoint third order generalization of the well-known problem describing the vibrational modes of a string with nonhomogeneous mass density. This fact lies at the core of our investigations of the inverse spectral problem.

3.1 Liouville transformation to a finite interval

A Liouville transformation is a change of both dependent and independent variables in order to bring an ODE to its simplest form. The Liouville transformation that we will use here was inspired by a similar transformation [2] relating the spectral problem for the Camassa–Holm equation to the string equation $\phi_{yy}(y) = z g(y) \phi(y)$ with Dirichlet boundary conditions $\phi(\pm 1) = 0$. This self-adjoint spectral problem, which we briefly review in Appendix A, is fundamental in the inverse scattering approach to finding multi-peakon solutions of the CH equation [3, 2]. As we will see, for the DP equation it is the non-selfadjoint “cubic string” that plays the corresponding role.

To get an idea of how the transformation was found, consider the DP spectral problem (2.31) in the simplest possible case $m(x) = 0$, and leave aside the boundary conditions for a moment. Then the equation is just $(\partial_x - \partial_x^3)\psi(x) = 0$ and the solution is

$$\begin{aligned} \psi(x) &= Ae^x + B + Ce^{-x} \\ &= \frac{(e^x + 1)^2}{2e^x} \left[\frac{A}{2} \left(\frac{2e^x}{e^x + 1} \right)^2 + \frac{B}{2} \frac{4e^x}{(e^x + 1)^2} + \frac{C}{2} \left(\frac{2}{e^x + 1} \right)^2 \right] \\ &= \frac{2}{1 - y^2} \left[A \frac{(1 + y)^2}{2} + B \frac{(1 + y)(1 - y)}{2} + C \frac{(1 - y)^2}{2} \right], \end{aligned} \quad (3.1)$$

where $y = \tanh(x/2) = \frac{e^x - 1}{e^x + 1}$. This change of variables maps $x \in \mathbf{R}$ to $y \in (-1, 1)$, and the inverse transformation is $x = \log \frac{1+y}{1-y}$. The expression in brackets, call it $\phi(y)$, is a quadratic polynomial and thus satisfies $-\partial_y^3 \phi = 0$; the first order term in the operator $\partial_x - \partial_x^3$ has been removed by the transformation.

In the general case, the following holds.

Theorem 3.1. *Under the change of variables*

$$y = \tanh \frac{x}{2}, \quad \psi(x) = \frac{2\phi(y)}{1-y^2}, \quad (3.2)$$

the DP spectral problem (2.31) is equivalent to the cubic string problem

$$\begin{aligned} -\phi_{yyy}(y) &= z g(y) \phi(y) \quad \text{for } y \in (-1, 1), \\ \phi(-1) &= \phi_y(-1) = 0, \\ \phi(1) &= 0, \end{aligned} \quad (3.3)$$

where

$$\left(\frac{1-y^2}{2} \right)^3 g(y) = m(x). \quad (3.4)$$

In the discrete case, when $m(x) = 2 \sum_1^n m_i \delta(x - x_i)$, equation (3.4) should be interpreted as

$$g(y) = \sum_{i=1}^n g_i \delta(y - y_i), \quad \text{where } y_i = \tanh \frac{x_i}{2}, \quad g_i = \frac{8m_i}{(1-y_i^2)^2}. \quad (3.5)$$

Proof. The first part is a straightforward computation with the chain rule, using $\frac{dx}{dy} = \frac{2}{1-y^2}$. In particular, the equivalence of the boundary conditions follows from

$$\begin{aligned} \phi(\pm 1) &= \lim_{x \rightarrow \pm \infty} \frac{2e^x}{(e^x + 1)^2} \psi(x), \\ \phi_y(-1) &= \lim_{x \rightarrow -\infty} \left(\psi_x(x) + \frac{1-e^x}{1+e^x} \psi(x) \right). \end{aligned}$$

For the discrete case, $\delta(x - x_i)dx = \delta(y - y_i)dy$, hence

$$\delta(x - x_i) = \frac{\delta(y - y_i)}{\frac{dx}{dy}(y_i)}$$

and the statement follows. \square

By analogy with the ordinary discrete string, we will refer to the quantities g_i as point masses at the positions y_i , although physically speaking this terminology is perhaps not justified. We also define $y_0 = -1$ and $y_{n+1} = 1$, which is consistent with our convention that $x_0 = -\infty$ and $x_{n+1} = \infty$.

Remark 3.2. For comparison, we remark that the Liouville transformation $y = \tanh x$, $\psi(x) = \phi(y)/\sqrt{1-y^2}$ used in the CH case [2] maps $\psi(x) = Ae^x + Be^{-x} \in \ker(\partial_x^2 - 1)$ to $\phi(y) = A(1+y) + B(1-y) \in \ker(\partial_y^2)$.

Since the adjoint of (3.3) involves two boundary conditions at the right endpoint and one at the left endpoint, (3.3) is not a selfadjoint problem. Hence there is no *a priori* reason to suspect that the eigenvalues need to be real. However, this follows in the discrete case immediately from Theorems 3.1 and 2.12.

Theorem 3.3. *The discrete cubic string problem (3.3), (3.5), with all $g_i > 0$, has n distinct positive eigenvalues*

$$0 < \lambda_1 < \cdots < \lambda_n.$$

In the general case, when the mass distribution $g(y)$ is not discrete, there is an infinite sequence of eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots,$$

which we will prove in Section 3.4 using the beautiful theory of oscillatory kernels developed by Gantmacher and Krein.

3.2 The discrete cubic string wavefunction $\phi(y; z)$

Because of Theorem 3.1, all notions defined in the context of the peakon spectral problem (2.31) have counterparts in the cubic string setting. The DP equation induces an isospectral deformation of the cubic string, but we mostly consider a fixed time t and suppress the time dependence in the notation.

The DP wavefunction $\psi(x; z)$ of Definition 2.6 is mapped by the Liouville transformation (3.2) to the *cubic string wavefunction* $\phi(y; z)$, which by definition is the solution of the initial value problem

$$\begin{aligned} \phi_{yyy}(y) + z g(y) \phi(y) &= 0 \quad \text{for } y \in (-1, 1), \\ \phi(-1) &= \phi_y(-1) = 0, \\ \phi_{yy}(-1) &= 1. \end{aligned} \tag{3.6}$$

Again, we are mainly interested in the discrete case

$$\begin{aligned} g(y) &= \sum_1^n g_i \delta(y - y_i), \quad g_i > 0, \\ -1 &= y_0 < y_1 < \cdots < y_n < y_{n+1} = +1. \end{aligned}$$

From (3.1) it is clear that the peakon wavefunction, piecewise given by

$$\psi(x; z) = A_k(z)e^x + B_k(z) + C_k(z)e^{-x}, \quad x \in (x_k, x_{k+1})$$

according to (2.20), corresponds to the *discrete cubic string wavefunction*

$$\phi(y; z) = A_k(z) \frac{(1+y)^2}{2} + B_k(z) \frac{1-y^2}{2} + C_k(z) \frac{(1-y)^2}{2}, \quad y \in (y_k, y_{k+1}), \tag{3.7}$$

which is piecewise a quadratic polynomial in y . We will denote by

$$l_k = y_{k+1} - y_k \tag{3.8}$$

the length of the k th interval (y_k, y_{k+1}) , for $k = 0, \dots, n$. The coefficients A_k , B_k , C_k in (3.7) are the same as before and are given by (2.26). In particular,

$A_0 = 1$ and $B_0 = C_0 = 0$, so that $\phi(y; z) = \frac{1}{2}(1+y)^2$ for $-1 \leq y \leq 1$. However, we will not make much use of these coefficients from now on; instead we will keep track of $\phi(y; z)$ through the values of the function and its derivatives at the points y_k . We will use the vector notation

$$\Phi(y; z) = \begin{pmatrix} \phi(y; z) \\ \phi_y(y; z) \\ \phi_{yy}(y; z) \end{pmatrix},$$

sometimes omitting z to simplify the notation.

By (3.6), $\phi_{yyy}(y)$ is singular (proportional to the delta function) at the points y_1, \dots, y_n , and zero elsewhere. Thus we require $\phi(y)$ and $\phi_y(y)$ to be continuous, and $\phi_{yy}(y)$ to be piecewise constant with jump discontinuities at y_1, \dots, y_n :

$$\phi_{yy}(y_k+) - \phi_{yy}(y_k-) = -z g_k \phi(y_k). \quad (3.9)$$

These jump conditions can be expressed as

$$\Phi(y_k+) = G_k(z) \Phi(y_k-), \quad \text{where} \quad G_k(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -z g_k & 0 & 1 \end{pmatrix}. \quad (3.10)$$

Further,

$$\phi(y) = \phi(y_k) + \phi_y(y_k)(y - y_k) + \phi_{yy}(y_k+) \frac{(y - y_k)^2}{2}$$

for $y_k \leq y \leq y_{k+1}$. Evaluating this expression and its derivatives at $y = y_{k+1}$ tells us how Φ propagates from one jump to the next:

$$\Phi(y_{k+1}-) = L_k \Phi(y_k+), \quad \text{where} \quad L_k = \begin{pmatrix} 1 & l_k & l_k^2/2 \\ 0 & 1 & l_k \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.11)$$

Clearly, the vectors $\Phi(y_k \pm; z)$ will consist of polynomials in z (with coefficients that depend on g_i 's and l_i 's). The degrees are as follows, for $k \geq 1$: as $z \rightarrow \infty$,

$$\Phi(y_k-; z) = L_{k-1} G_{k-1}(z) \dots L_1 G_1(z) L_0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} O(z^{k-1}) \\ O(z^{k-1}) \\ O(z^{k-1}) \end{pmatrix} \quad (3.12)$$

and

$$\Phi(y_k+; z) = G_k(z) \Phi(y_k-; z) = \begin{pmatrix} O(z^{k-1}) \\ O(z^{k-1}) \\ O(z^k) \end{pmatrix}. \quad (3.13)$$

It is immediate that $G_k(z)^{-1} = G_k(-z)$ and $L_k(l_k)^{-1} = L_k(-l_k)$. Some slightly less obvious relations, which are crucial in the solution of the inverse spectral problem, concern the transposed inverses:

$$(L_k^{-1})^t = J L_k J \quad \text{and} \quad (G_k(z)^{-1})^t = J G_k(-z) J, \quad (3.14)$$

where

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = J^{-1}. \quad (3.15)$$

3.3 Weyl functions

The eigenvalues of the cubic string problem (3.3) are the zeros of the function $\phi(1; z)$, where $\phi(y; z)$ is the cubic string wavefunction defined in the previous section as the solution of the initial value problem (3.6). The extended spectral data are encoded in a pair of *Weyl functions*.

Definition 3.4. The Weyl functions for the cubic string are

$$W(z) = \frac{\phi_y(1; z)}{\phi(1; z)}, \quad Z(z) = \frac{\phi_{yy}(1; z)}{\phi(1; z)}, \quad (3.16)$$

where $\phi(y; z)$ is the cubic string wavefunction.

Theorem 3.5. In the discrete case $g(y) = \sum_{k=1}^n g_k \delta(y - y_k)$, $g_k > 0$, the Weyl functions admit the partial fraction decompositions

$$\frac{W(z)}{z} = \frac{1}{z} + \sum_{k=1}^n \frac{b_k}{z - \lambda_k} = \sum_{k=0}^n \frac{b_k}{z - \lambda_k}, \quad (3.17)$$

$$\frac{Z(z)}{z} = \frac{1/2}{z} + \sum_{k=1}^n \frac{c_k}{z - \lambda_k} = \sum_{k=0}^n \frac{c_k}{z - \lambda_k}, \quad (3.18)$$

where $\{\lambda_k, b_k, c_k\}_{k=0}^n$ constitute the extended spectral data of the corresponding peakon spectral problem (see Theorem 3.1 and Definition 2.14).

Proof. In the discrete case, $\phi(1; z)$ is an n th degree polynomial in z . The eigenvalues $\lambda_1, \dots, \lambda_n$, which are simple and positive by Theorem 3.3, are the zeros of $\phi(1; z)$. This shows that $W(z)/z$ and $Z(z)/z$ have only simple poles, precisely at $z = 0 = \lambda_0$ and at each eigenvalue $z = \lambda_k$. To compute the residues at $z = 0$, we use that $\phi(y; 0) = \frac{1}{2}(1 + y)^2$, which gives $b_0 = \frac{\phi_y(1; 0)}{\phi(1; 0)} = \frac{2}{2}$ and $c_0 = \frac{\phi_{yy}(1; 0)}{\phi(1; 0)} = \frac{1}{2}$ in agreement with our definitions.

In the rightmost interval $y_n < y < 1$ we have, by (3.7),

$$\phi(y; z) = A(z) \frac{(1 + y)^2}{2} + B(z) \frac{1 - y^2}{2} + C(z) \frac{(1 - y)^2}{2},$$

hence

$$\frac{W(z)}{z} = \frac{\phi_y(1; z)}{z \phi(1; z)} = \frac{2A(z) - B(z)}{2z A(z)} = \frac{1}{z} - \frac{B(z)}{2z A(z)}$$

and

$$\frac{Z(z)}{z} = \frac{\phi_{yy}(1; z)}{z \phi(1; z)} = \frac{A(z) - B(z) + C(z)}{2z A(z)} = \frac{1/2}{z} + \frac{C(z) - B(z)}{2z A(z)}.$$

Comparison with (2.33) finishes the proof. \square

Corollary 3.6. *The second Weyl function $Z(z)$ is determined by the first Weyl function $W(z)$ through equation (2.36) expressing c_k in terms of λ_k 's and b_k 's.*

This fact, which is absolutely fundamental to the solution of the inverse spectral problem, is far from obvious from the definition of the Weyl functions in terms of a given discrete cubic string. In fact, the proof of (2.36) in Section 2.7 was only made possible by knowing how the spectral data change under the peakon evolution. In other words, in order to prove Corollary 3.6 we needed to know about the isospectral deformation of the cubic string induced by the DP equation.

3.4 Gantmacher–Krein theory and the cubic string

Before we turn to the inverse spectral problem, we would like to show how the cubic string fits into the Gantmacher–Krein theory of oscillatory kernels [12]. We begin by collecting a few facts about three classes of matrices with nonnegative entries, see e.g [11] or [10].

Definition 3.7. An $n \times n$ matrix A is called *totally positive* (*totally nonnegative*) if every minor of A is positive (nonnegative). A totally nonnegative matrix A is called *oscillatory* if some power of it is totally positive.

Oscillatory matrices can be thought of as being somewhere between totally nonnegative and totally positive matrices. The most pertinent property of these classes of matrices is the following.

Theorem 3.8. *All eigenvalues of a totally positive matrix are positive and of algebraic multiplicity one. All eigenvalues of a totally nonnegative matrix are nonnegative, but in general of arbitrary multiplicity.*

Corollary 3.9. *All eigenvalues of an oscillatory matrix are positive and of algebraic multiplicity one.*

Definition 3.10. Suppose I is an open interval of \mathbf{R} . A (necessarily positive) continuous function $K(y, s)$, $y, s \in I$ is called *oscillatory* if for every choice of points $y_1 < y_2 < \dots < y_n \in I$ the matrix $[K(y_i, y_j)]_{i,j=1,\dots,n}$ is oscillatory.

Consider the integral equation

$$\phi(y) = z \int_I K(y, s) \phi(s) d\sigma(s), \quad (3.19)$$

where the integral is taken in the sense of Stieltjes, with σ a non-decreasing function on I . In the above, z plays the role of an eigenvalue and the sought solution ϕ is the corresponding eigenfunction. The central result of the theory developed by Gantmacher and Krein is the following theorem.

Theorem 3.11 (Gantmacher–Krein). *If the kernel K of the integral equation (3.19) is oscillatory, then:*

1. The eigenvalues are all positive and simple: $0 < \lambda_1 < \lambda_2 < \dots$. If the function $\sigma(s)$ has only a finite number n of points of growth, then there are n eigenvalues. Otherwise, there are infinitely many eigenvalues.
2. The eigenfunction ϕ_1 corresponding to the smallest eigenvalue λ_1 has no zeros in the interval I .
3. For every $j > 1$, the eigenfunction ϕ_j corresponding to the j th eigenvalue λ_j has exactly $j - 1$ nodal points in I (i.e., it has $j - 1$ zeros and changes its sign at each).

The integral equation (3.19) occurs most naturally in the context of boundary value problems for ordinary differential operators.

Theorem 3.12 (Krein [14]). *Let $L(\phi(y)) = \sum_{k=0}^n l_k(y) \phi^{(k)}(y)$, where $l_n(y) > 0$, and consider the boundary value problem*

$$\begin{aligned} L(\phi) &= 0, \\ \phi(a) &= \phi'(a) = \dots = \phi^{(p-1)}(a) = 0, \\ \phi(b) &= \phi'(b) = \dots = \phi^{(q-1)}(b) = 0, \end{aligned}$$

where $p + q = n \geq 2$. Let $G(y, s)$ be the Green function corresponding to this boundary value problem and let $W(f_1, f_2, \dots, f_k)$ denote the Wronskian of the k times differentiable functions f_1, f_2, \dots, f_k . Then $(-1)^q G(y, s)$ is oscillatory if and only if there exist p solutions $\omega_1(y), \omega_2(y), \dots, \omega_p(y)$ of the boundary value problem

$$\begin{aligned} L(\phi) &= 0, \\ \phi(b) &= \phi'(b) = \dots = \phi^{(q-1)}(b) = 0 \end{aligned}$$

and q solutions $\omega_{p+1}(y), \omega_{p+2}(y), \dots, \omega_n(y)$ of the boundary value problem

$$\begin{aligned} L(\phi) &= 0, \\ \phi(a) &= \phi'(a) = \dots = \phi^{(p-1)}(a) = 0 \end{aligned}$$

such that, for all $y \in I$,

$$\omega_1 > 0, \quad W(\omega_1, \omega_2) > 0, \quad \dots, \quad W(\omega_1, \omega_2, \dots, \omega_n) > 0.$$

The cubic string equation (3.3) is a special case of this setup that corresponds to $p = 2, q = 1, L = d^3/dy^3$, and $I = (-1, 1)$. The Green function of the cubic string is the solution of

$$\begin{aligned} G_{yyy}(y, s) &= \delta(y - s), \\ G(-1, s) &= G_y(-1, s) = 0, \quad G(1, s) = 0, \end{aligned}$$

namely

$$G(y, s) = \begin{cases} -\frac{(s-1)^2(y+1)^2}{8}, & -1 \leq y \leq s \leq 1, \\ \frac{(s+1)^2(y-1)^2}{8} - \frac{(s^2-1)(y^2-1)}{4}, & -1 \leq s \leq y \leq 1. \end{cases} \quad (3.20)$$

Lemma 3.13. *Let $G(y, s)$ be the Green function (3.20) for the cubic string. Then $K(y, s) = -G(y, s)$ is oscillatory.*

Proof. By direct computation one verifies that $\omega_1(y) = 1 - y$, $\omega_2(y) = -(1 - y)^2$, and $\omega_3(y) = (1 + y)^2$ satisfy the conditions in Theorem 3.12. \square

The cubic string problem (3.3) is equivalent to the integral equation

$$\phi(y) = z \int_{-1}^1 K(y, s) \phi(s) d\sigma(s), \quad (3.21)$$

where $K(y, s) = -G(y, s)$ and $d\sigma(s) = g(s) ds$. In this formulation we can include any kind of (positive) mass distribution, by letting the non-decreasing function $\sigma(s)$ equal the accumulated mass in the interval $-1 \leq y \leq s$. Since K is oscillatory, Theorem 3.11 gives the following more general version of Theorem 3.3, valid not only in the discrete case.

Theorem 3.14. *The spectrum of the cubic string problem (3.21) is positive and simple: $0 < \lambda_1 < \lambda_2 < \dots$.*

Theorem 3.11 also provides an alternative proof of the positivity of the b_k 's.

Theorem 3.15. *The residues b_k in the Weyl function $W(z)$ of the discrete cubic string are positive. (See equation (3.17).)*

Proof. We have

$$b_k = \operatorname{res}_{z=\lambda_k} \frac{\phi_y(1; z)}{z \phi(1; z)} = \frac{-\phi_y(1; \lambda_k)}{2 \prod_{j \neq k} (1 - \lambda_k / \lambda_j)}, \quad (3.22)$$

where $\phi(y; z) = 2 \prod_{j=1}^n (1 - z / \lambda_j)$ is the solution to the cubic string initial value problem with $\phi(-1; z) = \phi_y(-1; z) = 0$ and $\phi_{yy}(1; z) = 1$. In other words, $\phi(y; \lambda_k)$ is the k th eigenfunction of the cubic string boundary value problem, and as such it has $k - 1$ nodal points according to Theorem 3.11. It follows that $(-1)^k \phi_y(1; \lambda_k) \geq 0$. Suppose equality holds. Then $\phi(y; \lambda_k)$ is simultaneously an eigenfunction of our usual cubic string and of the mirrored cubic string ($p = 1, q = 2$) with boundary conditions $\phi(1; z) = \phi_y(1; z) = \phi(-1; z) = 0$. This is a contradiction, since it follows immediately from Theorems 3.11 and 3.12 that the mirrored cubic string has negative eigenvalues. Hence, $(-1)^k \phi_y(1; \lambda_k) > 0$. And since precisely $k - 1$ factors in the denominator of (3.22) are negative, it follows that $b_k > 0$. \square

4 Inverse problem for the discrete cubic string

Our aim in this section is to give an explicit solution of the inverse spectral problem for the discrete cubic string (3.3), (3.5). This will also provide us with what is needed to complete the proof of Theorem 2.22 regarding the corresponding inverse spectral problem for DP peakons.

The inverse problem for the ordinary (not cubic) discrete string plays a fundamental role in solving the peakon problem for the Camassa–Holm equation [3, 2]. For the reader’s convenience, we sketch the Stieltjes–Krein solution to that inverse problem in Appendix A, with emphasis on the structures common to both inverse problems.

4.1 General setup

We begin by stating the problem precisely.

Definition 4.1 (Inverse problem). Consider a discrete cubic string specified by a positive measure $g(y) = \sum_{i=1}^n g_i \delta_{y_i}(y)$ whose support satisfies $-1 < y_1 < y_2 < \dots < y_n < 1$. The inverse spectral problem is to determine the measure $g(y)$ given the Weyl function $W(z)$ of the cubic string:

$$\frac{W(z)}{z} = \sum_{i=0}^n \frac{b_i}{z - \lambda_i} = \int \frac{d\mu(\lambda)}{z - \lambda}$$

where

$$\mu = \sum_{i=0}^n b_i \delta_{\lambda_i},$$

$b_0 = 1$, $b_k > 0$, and $0 = \lambda_0 < \lambda_1 < \dots < \lambda_n$,

Recall that by Definition 3.4 the Weyl functions are

$$W(z) = \frac{\phi_y(1; z)}{\phi(1; z)}, \quad Z(z) = \frac{\phi_{yy}(1; z)}{\phi(1; z)},$$

where, according to (3.6), (3.10), (3.11),

$$\Phi(1; z) = \begin{pmatrix} \phi(1; z) \\ \phi_y(1; z) \\ \phi_{yy}(1; z) \end{pmatrix} = L_n G_n(z) L_{n-1} G_{n-1}(z) \cdots L_1 G_1(z) L_0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Also remember that Z is determined by W (Corollary 3.6).

The analogy with the case of Stieltjes continued fraction (see Appendix A) suggests that the 3×3 matrices

$$\begin{aligned} a^{(1)}(z) &= L_n \\ a^{(2)}(z) &= L_n G_n(z) \\ a^{(3)}(z) &= L_n G_n(z) L_{n-1} \\ a^{(4)}(z) &= L_n G_n(z) L_{n-1} G_{n-1}(z) \\ &\vdots \end{aligned}$$

might contain convergent-like quantities from which one can construct successively better rational approximations to $W(z)$ and $Z(z)$. Likewise, from the vectors

$$\begin{aligned}\Phi(y_0+; z) &= (0, 0, 1)^t \\ \Phi(y_1-; z) &= L_0 (0, 0, 1)^t \\ \Phi(y_1+; z) &= G_1(z) L_0 (0, 0, 1)^t \\ \Phi(y_2-; z) &= L_1 G_1(z) L_0 (0, 0, 1)^t \\ \Phi(y_2+; z) &= G_2(z) L_1 G_1(z) L_0 (0, 0, 1)^t \\ &\vdots\end{aligned}$$

one should be able to form remainder-like quantities. This is indeed the case, as we shall soon show.

In contrast to the classical Stieltjes theory, it is not necessary to consider both the even and the odd case. This is because the matrices $a^{(2k)}(z)$ and $a^{(2k-1)}(z)$ differ only in the first column, and it turns out that the information that we want to extract can be found in their common second column. We choose here to study the even case, where the starting point is the relation

$$\Phi(1; z) = a^{(2k)}(z) \Phi(y_{n-k+1}-; z). \quad (4.1)$$

We will for the most part regard $k \in \{1, \dots, n\}$ as fixed (but arbitrary), and denote the matrix $a^{(2k)}(z)$ simply by $a(z)$ or a . We collect some basic facts about it in the following lemma. As in Section 2.8, it is convenient to let $k' = n + 1 - k$.

Lemma 4.2. *The matrix*

$$a(z) = a^{(2k)}(z) = L_n G_n(z) L_{n-1} G_{n-1}(z) \cdots L_{k'} G_{k'}(z) \quad (4.2)$$

satisfies $\det a(z) = 1$. The entries in its first column are polynomials in z of degree k , while the remaining entries are polynomials of degree $k - 1$. In the second column we note in particular that

$$a_{12}(z) = l_{k'} \left(\prod_{i=k'+1}^n \frac{-g_i l_i^2}{2} \right) z^{k-1} + \dots + \left(\sum_{i=k'}^n l_i \right) \quad (4.3)$$

and

$$a_{22}(0) = 1, \quad a_{32}(0) = 0. \quad (4.4)$$

All 2×2 minors of $a(z)$ are polynomials of degree at most k .

Proof. The matrices $G_i(z)$ and L_i have determinant one, hence so does $a(z)$. Since $G_i(z)$ is linear in z , it is clear that $a(z)$ is a matrix polynomial in z of degree k . However, the second and third columns in $a(z)$ are unaffected

by the last factor $G_{k'}(z)$, and are consequently only of degree $k - 1$. To extract the leading coefficient, write $G_i(z) = I + z(-g_i)(0, 0, 1)^t(1, 0, 0)$ and use $(1, 0, 0)L_i(0, 0, 1)^t = l_i^2/2$. The constant term is

$$a(0) = L_n L_{n-1} \cdots L_{k'} = \begin{pmatrix} 1 & \sum_{i=k'}^n l_i & \left(\sum_{i=k'}^n l_i\right)^2/2 \\ 0 & 1 & \sum_{i=k'}^n l_i \\ 0 & 0 & 1 \end{pmatrix}.$$

Finally, because $a(z)$ has determinant one, its adjoint equals $a(z)^{-1}$, which is of degree k since $G_i(z)^{-1} = G_i(-z)$, thus implying that all 2×2 minors of $a(z)$ are polynomials of degree at most k . \square

Returning now to equation (4.1), we divide it by $\phi(1; z)$ and let $\Phi(y_{k'} -; z) = (p_k, q_k, r_k)^t$ to obtain

$$\begin{pmatrix} 1 \\ W(z) \\ Z(z) \end{pmatrix} = \frac{1}{\phi(1; z)} a^{(2k)}(z) \begin{pmatrix} p_k(z) \\ q_k(z) \\ r_k(z) \end{pmatrix}. \quad (4.5)$$

Suppressing the dependence on z and k in order to simplify the notation, we find

$$\frac{W}{1} = \frac{a_{21} p + a_{22} q + a_{23} r}{a_{11} p + a_{12} q + a_{13} r} = \frac{a_{21} R + a_{22} + a_{23} \hat{R}}{a_{11} R + a_{12} + a_{13} \hat{R}}, \quad (4.6)$$

where $R = p/q$ and $\hat{R} = r/q$. Similarly,

$$\frac{Z}{1} = \frac{a_{31} R + a_{32} + a_{33} \hat{R}}{a_{11} R + a_{12} + a_{13} \hat{R}}. \quad (4.7)$$

The quantities R and \hat{R} , or more precisely

$$R_{2k} = \frac{p_k}{q_k} = \frac{\phi(y_{k'}; z)}{\phi_y(y_{k'}; z)} \quad \text{and} \quad \hat{R}_{2k} = \frac{r_k}{q_k} = \frac{\phi_{yy}(y_{k'} -; z)}{\phi_y(y_{k'}; z)}, \quad (4.8)$$

are the analogs of *remainders* referred to earlier. By formally setting the remainders equal to zero in (4.6) and (4.7) we get the analogs of *convergents*, $W \approx a_{22}/a_{12}$ and $Z \approx a_{32}/a_{12}$, with the following order of approximation:

Theorem 4.3. *As $z \rightarrow \infty$,*

$$W(z) = \frac{a_{22}^{(2k)}(z)}{a_{12}^{(2k)}(z)} + O\left(\frac{1}{z^{k-1}}\right), \quad (4.9a)$$

$$Z(z) = \frac{a_{32}^{(2k)}(z)}{a_{12}^{(2k)}(z)} + O\left(\frac{1}{z^{k-1}}\right). \quad (4.9b)$$

Proof. It follows from (4.6) that

$$W - \frac{a_{22}}{a_{12}} = \frac{\frac{a_{12}a_{21} - a_{11}a_{22}}{a_{12}} R + \frac{a_{12}a_{23} - a_{13}a_{22}}{a_{12}} \widehat{R}}{a_{11} R + a_{12} + a_{13} \widehat{R}}.$$

By (3.12), p , q , and r are all of degree $k' - 1$, hence $R = p/q = O(1)$ and $\widehat{R} = r/q = O(1)$. From this, and from Lemma 4.2, we see that the right-hand side equals

$$\frac{\frac{O(z^k)}{O(z^{k-1})}O(1) + \frac{O(z^k)}{O(z^{k-1})}O(1)}{O(z^k)O(1) + O(z^{k-1}) + O(z^{k-1})O(1)} = O\left(\frac{1}{z^{k-1}}\right).$$

The proof for Z is similar. □

Theorem 4.4. *Let*

$$W^*(z) = -W(-z), \quad Z^*(z) = Z(-z). \quad (4.10)$$

Then, with $a = a^{(2k)}(z)$,

$$Z^* a_{12} + W^* a_{22} + a_{32} = O\left(\frac{1}{z^k}\right), \quad \text{as } z \rightarrow \infty. \quad (4.11)$$

Proof. From (4.5) we have

$$a(z)^{-1} \begin{pmatrix} 1 \\ W(z) \\ Z(z) \end{pmatrix} = \frac{1}{\phi(1; z)} \Phi(y_{k'} -; z) = O\left(\frac{1}{z^k}\right).$$

Now recall (3.14), which implies that $(a(z)^{-1})^t = Ja(-z)J$, where $J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. (Note that $J = J^{-1} = J^t$.) Hence

$$Ja(-z)^t J \begin{pmatrix} 1 \\ W(z) \\ Z(z) \end{pmatrix} = O\left(\frac{1}{z^k}\right),$$

which upon changing z to $-z$ and multiplying both sides by J yields

$$a(z)^t \begin{pmatrix} Z(-z) \\ -W(-z) \\ 1 \end{pmatrix} = O\left(\frac{1}{z^k}\right),$$

the second row of which is precisely (4.11). □

4.2 An approximation problem

Lemma 4.2, Theorem 4.3, and Theorem 4.4 imply that the entries in the second column of $a = a^{(2k)}(z)$,

$$(Q, P, \hat{P}) = (a_{12}, a_{22}, a_{32}),$$

satisfy the following approximation problem:

Definition 4.5 (Padé-like approximation problem). Let the functions W and Z be given by

$$\begin{aligned} \frac{W(z)}{z} &= \sum_{k=0}^n \frac{b_k}{z - \lambda_k} = \sum_{j=0}^{\infty} \frac{\beta_j}{z^{j+1}}, \\ \frac{Z(z)}{z} &= \sum_{k=0}^n \frac{c_k}{z - \lambda_k} = \sum_{j=0}^{\infty} \frac{\gamma_j}{z^{j+1}}, \end{aligned} \quad (4.12)$$

where

$$\beta_j = \sum_{k=0}^n b_k \lambda_k^j, \quad \gamma_j = \sum_{k=0}^n c_k \lambda_k^j, \quad (4.13)$$

and the c_k 's depend on b_k 's and λ_k 's as in Corollary 2.16. For a given integer $1 \leq k \leq n$, we seek three polynomials (Q, P, \hat{P}) of degree $k-1$ satisfying the following conditions:

1. (Approximation)

$$W = \frac{P}{Q} + O\left(\frac{1}{z^{k-1}}\right), \quad Z = \frac{\hat{P}}{Q} + O\left(\frac{1}{z^{k-1}}\right) \quad (z \rightarrow \infty).$$

2. (Symmetry)

$$Z^* Q + W^* P + \hat{P} = O\left(\frac{1}{z^k}\right) \quad (z \rightarrow \infty),$$

where $W^*(z) = -W(-z)$ and $Z^*(z) = Z(-z)$.

3. (Normalization at $z = 0$)

$$P(0) = 1, \quad \hat{P}(0) = 0.$$

Remark 4.6. The approximation condition is similar to classical Padé–Hermite approximation [17], where one approximates two (independently chosen) functions to order $O(1/z^{2(k-1)})$ using a common denominator of degree $k-1$. Since in our case we only have $O(1/z^{k-1})$, there is clearly not enough information in the approximation condition alone to uniquely determine the denominator $Q(z)$. However, our W and Z are not independent of each other, and the additional symmetry condition will eventually lead to a unique $Q(z)$.

If we introduce *spectral measures*

$$\mu = \sum_{k=0}^n b_k \delta_{\lambda_k} \quad \text{and} \quad \nu = \sum_{k=0}^n c_k \delta_{\lambda_k}, \quad (4.14)$$

then

$$\beta_j = \sum_{k=0}^n b_k \lambda_k^j = \int x^j d\mu(x) \quad \text{and} \quad \gamma_j = \sum_{k=0}^n c_k \lambda_k^j = \int x^j d\nu(x) \quad (4.15)$$

are the moments of μ and ν , respectively. By Corollary 2.16, ν is determined by μ . In fact, (2.36) can be expressed as

$$\int f(x) d\nu(x) = \iint \frac{x f(x)}{x+y} d\mu(x) d\mu(y) = \iint \frac{y f(y)}{x+y} d\mu(x) d\mu(y). \quad (4.16)$$

In particular,

$$\gamma_j = \iint \frac{x^{j+1}}{x+y} d\mu(x) d\mu(y) = \iint \frac{y^{j+1}}{x+y} d\mu(x) d\mu(y). \quad (4.17)$$

Lemma 4.7. *The relation (4.16) between μ and ν implies that W and Z satisfy the constraint*

$$Z(z) + Z^*(z) + W^*(z)W(z) = 0.$$

Proof. Expressing W and Z in terms of the spectral measures we obtain

$$W(z) = \int \frac{z}{z-x} d\mu(x), \quad W^*(z) = - \int \frac{z}{z+y} d\mu(y),$$

and

$$\begin{aligned} Z(z) &= \int \frac{z}{z-x} d\nu(x) = \iint \frac{zx}{z-x} \frac{1}{x+y} d\mu(x) d\mu(y), \\ Z^*(z) &= \int \frac{z}{z+x} d\nu(x) = \iint \frac{zy}{z+y} \frac{1}{x+y} d\mu(x) d\mu(y), \end{aligned}$$

from which the identity in question follows immediately. \square

For the purposes of this approximation problem, we can consider all functions as being formal Laurent series with finitely many positive powers. Let Π_{\pm} and Π_{\pm}^0 denote the following projection operators on finite dimensional subspaces (we regard k as fixed, and omit dependence on k in the notation):

$$\begin{aligned} \Pi_+ \left(\sum_{i=-\infty}^N a_i z^i \right) &= \sum_{i=1}^{k-1} a_i z^i, & \Pi_+^0 \left(\sum_{i=-\infty}^N a_i z^i \right) &= \sum_{i=0}^{k-1} a_i z^i, \\ \Pi_- \left(\sum_{i=-\infty}^N a_i z^i \right) &= \sum_{i=-(k-1)}^0 a_i z^i. \end{aligned}$$

Let M_f denote multiplication by $f(z)$, and define the truncated Hankel and Toeplitz operators

$$\begin{aligned} H_f &= \Pi_- \circ M_f \circ \Pi_+^0 : \text{span}\{z^0, \dots, z^{k-1}\} \rightarrow \text{span}\{z^{-k+1}, \dots, z^0\}, \\ T_f &= \Pi_+ \circ M_f \circ \Pi_+^0 : \text{span}\{z^0, \dots, z^{k-1}\} \rightarrow \text{span}\{z^1, \dots, z^{k-1}\}. \end{aligned} \quad (4.18)$$

Moreover, let

$$\iota : \text{span}\{z^1, \dots, z^{k-1}\} \rightarrow \text{span}\{z^0, z^1, \dots, z^{k-1}\}$$

be the natural inclusion.

Theorem 4.8. *The polynomials $Q(z)$, $P(z)$, $\hat{P}(z)$ solve the approximation problem of Definition 4.5 if and only if*

$$P = 1 + \Pi_+ WQ, \quad \hat{P} = \Pi_+ ZQ, \quad (4.19)$$

$$[H_{Z^*} + H_{W^*} \circ \iota \circ T_W] Q = -\Pi_- W^*. \quad (4.20)$$

Equation (4.20) is equivalent to the linear system

$$\begin{pmatrix} \delta_{00} & \delta_{01} & \dots & \delta_{0,k-1} \\ \delta_{10} & \delta_{11} & \dots & \delta_{1,k-1} \\ \vdots & \vdots & & \vdots \\ \delta_{k-1,0} & \delta_{k-1,1} & \dots & \delta_{k-1,k-1} \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ \vdots \\ q_{k-1} \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{k-1} \end{pmatrix} \quad (4.21)$$

for the unknown coefficients in $Q(z) = \sum_{i=0}^{k-1} q_i z^i$, where

$$\delta_{ab} = \iint \frac{x^{a+1} y^b}{x+y} d\mu(x) d\mu(y). \quad (4.22)$$

Proof. According to the approximation condition, the functions $WQ - P$ and $ZQ - \hat{P}$ contain no positive powers of z , hence their Π_+ projections are zero. This, together with the normalization conditions, is equivalent to (4.19) (which shows that P and \hat{P} are uniquely determined by Q). Inserting (4.19) into the symmetry condition yields

$$Z^* Q + W^* (1 + \Pi_+ WQ) + \Pi_+ ZQ = O\left(\frac{1}{z^k}\right). \quad (4.23)$$

The coefficients of the positive powers z^1, \dots, z^{k-1} appearing in (4.23) vanish identically regardless of Q ; indeed, using $(\Pi_+ - I)Q = -q_0$, $\Pi_+ W^* = 0$, and Lemma 4.7, we obtain

$$\begin{aligned} \Pi_+ (Z^* Q + W^* (1 + \Pi_+ WQ) + \Pi_+ ZQ) &= \Pi_+ (Z + Z^* + W^* \Pi_+ W) Q \\ &= \Pi_+ (Z + Z^* + W^* (\Pi_+ - I + I) W) Q = \Pi_+ (Z + Z^* + W^* W) Q = 0. \end{aligned}$$

On the other hand, the vanishing of the coefficients of $z^{-(k-1)}, \dots, z^0$ in (4.23) is equivalent to the following condition on Q :

$$\Pi_- (Z^* Q + W^* (1 + \Pi_+ W Q) + \Pi_+ Z Q) = 0, \quad (4.24)$$

which is just (4.20).

To show equivalence of (4.20) and (4.21) we write out (4.20) in terms of matrices with respect to the standard bases ordered as in (4.18). Then Q is represented by the column vector $(q_0, \dots, q_{n-1})^t$, while

$$\begin{aligned} H_{Z^*} &= \left[(-1)^{a+b} \gamma_{a+b} \right]_{\substack{0 \leq a \leq k-1 \\ 0 \leq b \leq k-1}} = \begin{pmatrix} \gamma_0 & -\gamma_1 & \gamma_2 & \cdots \\ -\gamma_1 & \gamma_2 & & \\ \gamma_2 & & & \\ \vdots & & & \end{pmatrix}, \\ H_{W^*} \circ \iota &= \left[(-1)^{a+m+1} \beta_{a+m} \right]_{\substack{0 \leq a \leq k-1 \\ 1 \leq m \leq k-1}} = \begin{pmatrix} \beta_1 & -\beta_2 & \beta_3 & \cdots \\ -\beta_2 & \beta_3 & & \\ \beta_3 & & & \\ \vdots & & & \end{pmatrix}, \\ T_W &= \left[\beta_{b-m} \right]_{\substack{1 \leq m \leq k-1 \\ 0 \leq b \leq k-1}} = \begin{pmatrix} 0 & \beta_0 & \beta_1 & \beta_2 & \cdots \\ 0 & 0 & \beta_0 & \beta_1 & \ddots \\ 0 & 0 & 0 & \beta_0 & \ddots \\ \vdots & & & & \ddots \end{pmatrix}, \\ -\Pi_- W^* &= (\beta_0, -\beta_1, \beta_2, \dots, (-1)^{k-1} \beta_{k-1})^t. \end{aligned}$$

If we multiply out the matrices and change the sign of every second row, the system takes the form

$$\sum_{b=0}^{k-1} \left[(-1)^b \gamma_{a+b} - \sum_{m=1}^b (-1)^m \beta_{a+m} \beta_{b-m} \right] q_b = \beta_a, \quad 0 \leq a \leq k-1. \quad (4.25)$$

(When $b=0$, the inner sum is empty.) Upon using the spectral representations (4.15) and (4.17) of β 's and γ 's we arrive at the claimed integral representation (4.22) of the (a, b) entry δ_{ab} of the matrix of coefficients in (4.25):

$$\begin{aligned} \delta_{ab} &= \iint \left[(-1)^b \frac{x^{a+b+1}}{x+y} - \sum_{m=1}^b (-1)^m x^{a+m} y^{b-m} \right] d\mu(x) d\mu(y) \\ &= \iint \frac{x^{a+1} y^b}{x+y} d\mu(x) d\mu(y). \end{aligned}$$

□

To complete the picture, it remains to show that the system (4.21) has nonzero determinant, so that the solution to the approximation problem exists and is unique. This will be dealt with in the following sections.

Remark 4.9. The choice of convergents and remainders is not unique. For example, one could take the entries in the *first* column of $a^{(2k)}(z)$ as convergents, and instead of (4.6) write

$$W = \frac{a_{21} + a_{22} \frac{q}{p} + a_{13} \frac{r}{p}}{a_{11} + a_{12} \frac{q}{p} + a_{13} \frac{r}{p}} \approx \frac{a_{21}}{a_{11}},$$

where now q/p and r/p play the role of remainders, and the order of approximation can be shown to be $O(1/z^k)$; similarly with Z and equation (4.7). In this way one obtains an approximation problem which is similar to, but slightly different from, the one in Definition 4.5. When looking at the first column, the odd matrices $a^{(2k-1)}$ are different from the even ones, and this case provides yet another approximation problem of similar type. However, we are convinced that the problem we have chosen to focus on here is the simplest route to the solution of the inverse problem.

4.3 Determinants

From the expression (4.22) for δ_{ab} as a double integral, it is clear that the determinants that appear when one tries to solve the linear system (4.21) with Cramer's rule are somewhat reminiscent of the classical $k \times k$ Hankel determinant of moments,

$$\begin{vmatrix} \beta_0 & \beta_1 & \dots & \beta_{k-1} \\ \beta_1 & \beta_2 & \dots & \beta_k \\ \vdots & & & \vdots \\ \beta_{k-1} & \beta_k & \dots & \beta_{2k-2} \end{vmatrix} = \frac{1}{k!} \int_{\mathbf{R}^k} \Delta(x)^2 d\mu^k(x), \quad (4.26)$$

where $\Delta(x) = \Delta(x_1, \dots, x_k) = \prod_{i < j} (x_i - x_j)$. (For a proof of Heine's formula (4.26), see for instance [19, page 27] or [8, Proposition 3.8].)

The following lemma contains similar integral formulas for the present determinants. These formulas are valid for any measure, although our main interest here is of course in the discrete case when $\mu = \sum_{i=0}^n b_i \delta_{\lambda_i}$.

Lemma 4.10. *Suppose μ is a measure on \mathbf{R} such that the integrals*

$$\beta_a = \int x^a d\mu(x) \quad \text{and} \quad \delta_{ab} = \iint \frac{x^{a+1}y^b}{x+y} d\mu(x) d\mu(y) \quad (4.27)$$

are finite. Let

$$\begin{aligned} u_k &= \frac{1}{k!} \int_{\mathbf{R}^k} \frac{\Delta(x)^2}{\Gamma(x)} d\mu^k(x), \\ v_k &= \frac{1}{k!} \int_{\mathbf{R}^k} \frac{\Delta(x)^2}{\Gamma(x)} x_1 x_2 \dots x_k d\mu^k(x), \end{aligned} \quad (4.28)$$

where

$$\begin{aligned}\Delta(x) &= \Delta(x_1, \dots, x_k) = \prod_{i < j} (x_i - x_j), \\ \Gamma(x) &= \Gamma(x_1, \dots, x_k) = \prod_{i < j} (x_i + x_j).\end{aligned}\tag{4.29}$$

(When $k = 0$ or 1 , we let $\Delta(x) = \Gamma(x) = 1$. Also, $u_0 = v_0 = 1$.) Then for $k \geq 1$ the following $k \times k$ determinant formulas hold:

$$D_k := \begin{vmatrix} \delta_{00} & \delta_{01} & \delta_{02} & \dots & \delta_{0,k-1} \\ \delta_{10} & \delta_{11} & \delta_{12} & \dots & \delta_{1,k-1} \\ \delta_{20} & \delta_{21} & \delta_{22} & \dots & \delta_{2,k-1} \\ \vdots & \vdots & \vdots & & \vdots \\ \delta_{k-1,0} & \delta_{k-1,1} & \delta_{k-1,2} & \dots & \delta_{k-1,k-1} \end{vmatrix} = \frac{(u_k)^2}{2^k}, \tag{4.30}$$

$$D'_k := \begin{vmatrix} \beta_0 & \delta_{00} & \delta_{01} & \dots & \delta_{0,k-2} \\ \beta_1 & \delta_{10} & \delta_{11} & \dots & \delta_{1,k-2} \\ \beta_2 & \delta_{20} & \delta_{21} & \dots & \delta_{2,k-2} \\ \vdots & \vdots & \vdots & & \vdots \\ \beta_{k-1} & \delta_{k-1,0} & \delta_{k-1,1} & \dots & \delta_{k-1,k-2} \end{vmatrix} = \frac{u_k u_{k-1}}{2^{k-1}}, \tag{4.31}$$

and

$$D''_k := \begin{vmatrix} \beta_0 & \delta_{01} & \delta_{02} & \dots & \delta_{0,k-1} \\ \beta_1 & \delta_{11} & \delta_{12} & \dots & \delta_{1,k-1} \\ \beta_2 & \delta_{21} & \delta_{22} & \dots & \delta_{2,k-1} \\ \vdots & \vdots & \vdots & & \vdots \\ \beta_{k-1} & \delta_{k-1,1} & \delta_{k-1,2} & \dots & \delta_{k-1,k-1} \end{vmatrix} = \frac{u_k v_{k-1}}{2^{k-1}}. \tag{4.32}$$

Proof. We show only (4.31); the formulas (4.30) and (4.32) are proved similarly. Notationally it is slightly easier to work with the $(k+1) \times (k+1)$ determinant D'_{k+1} , so what we will actually prove is that

$$D'_{k+1} = \frac{u_k u_{k+1}}{2^k}. \tag{4.33}$$

Note first that our definition of $\Delta(x)$ is such that the Vandermonde determinant equals

$$\det(x_i^{j-1})_{i,j=1,\dots,k} = \prod_{i > j} (x_i - x_j) = \Delta(x_k, \dots, x_1) = (-1)^{\binom{k}{2}} \Delta(x_1, \dots, x_k).$$

In the final result the sign convention does not matter since only $\Delta(x)^2$ appears there, but we need to keep track of the correct signs during the proof. As regards $\Gamma(x)$, the order of the variables is of course immaterial.

Like in the proof of Heine's formula, we use different dummy variables in each column, in order to be able to pull integrals outside of the determinant

by multilinearity. We will need $2k + 1$ variables $(x_0, x_1, \dots, x_{2k})$, as follows. In the integrals β_a in the first column we call the integration variable x_0 , in the double integrals δ_{a0} in the second column we use x_1 and x_2 , in the double integrals δ_{a1} in the third column we use x_3 and x_4 , etc., with even-numbered variables replacing x and odd-numbered variables replacing y . Then we can pull out all integral signs, and also the factor $x_{2b-1}^{b-1} x_{2b} / (x_{2b-1} + x_{2b})$ from column $b + 1$, for $b = 1, \dots, k$. What remains is a Vandermonde determinant in the even-numbered variables, so

$$D'_{k+1} = \int_{\mathbf{R}^{2k+1}} \frac{(x_1^0 x_3^1 x_5^2 \cdots x_{2k-1}^{k-1})(x_2 x_4 \cdots x_{2k}) \Delta(x_{2k}, \dots, x_4, x_2, x_0)}{(x_1 + x_2)(x_3 + x_4) \cdots (x_{2k-1} + x_{2k})} d\mu^{2k+1}(x).$$

We now introduce an auxiliary variable $x_{-1} = 0$ in order to write the last expression in a more symmetric form:

$$D'_{k+1} = \int_{\mathbf{R}^{2k+1}} \frac{(x_1^0 x_3^1 x_5^2 \cdots x_{2k-1}^{k-1})(x_0 x_2 x_4 \cdots x_{2k}) \Delta(x_{2k}, \dots, x_4, x_2, x_0)}{(x_{-1} + x_0)(x_1 + x_2)(x_3 + x_4) \cdots (x_{2k-1} + x_{2k})} d\mu^{2k+1}(x).$$

The value of the original determinant is independent of the order in which we number the dummy variables, so the above expression is invariant under permutations of the variables. In particular, symmetrizing over the even-numbered variables doesn't change the value. That is, if S_{even} denotes the set of permutations of $\{0, 2, 4, \dots, 2k\}$, then

$$D'_{k+1} = \frac{1}{(k+1)!} \int_{\mathbf{R}^{2k+1}} (x_1^0 x_3^1 \cdots x_{2k-1}^{k-1})(x_0 x_2 x_4 \cdots x_{2k}) \Delta(x_{2k}, \dots, x_4, x_2, x_0) \\ \times \left(\sum_{\pi \in S_{\text{even}}} \frac{(\text{sgn } \pi)}{(x_{-1} + x_{\pi(0)})(x_1 + x_{\pi(2)})(x_3 + x_{\pi(4)}) \cdots (x_{2k-1} + x_{\pi(2k)})} \right) d\mu^{2k+1}(x).$$

The sum in parentheses is a Cauchy determinant, which by the well-known formula equals

$$\frac{\Delta(x_{-1}, x_1, x_3, \dots, x_{2k-1}) \Delta(x_0, x_2, x_4, \dots, x_{2k})}{\prod_{i=0}^k \prod_{j=0}^k (x_{2i-1} + x_{2j})} \\ = \frac{(-x_1)(-x_3) \cdots (-x_{2k-1}) \Delta(x_1, x_3, \dots, x_{2k-1}) \Delta(x_0, x_2, x_4, \dots, x_{2k})}{(x_0 x_2 x_4 \cdots x_{2k}) \prod_{i=1}^k \prod_{j=0}^k (x_{2i-1} + x_{2j})},$$

where the second line is a result of splitting off terms involving x_{-1} . It follows that

$$D'_{k+1} = \frac{1}{(k+1)!} \int_{\mathbf{R}^{2k+1}} (-1)^{\binom{k+1}{2} + k} (x_1 x_3 \cdots x_{2k-1}) (x_1^0 x_3^1 \cdots x_{2k-1}^{k-1}) \\ \times \frac{\Delta(x_1, x_3, \dots, x_{2k-1}) \Delta(x_0, x_2, \dots, x_{2k})^2}{\prod_{i=1}^k \prod_{j=0}^k (x_{2i-1} + x_{2j})} d\mu^{2k+1}(x).$$

Symmetrizing this over the odd-numbered variables we get yet another Vandermonde determinant $(-1)^{\binom{k}{2}} \Delta(x_1, x_3, \dots, x_{2k-1})$, hence

$$D'_{k+1} = \frac{1}{k!(k+1)!} \int_{\mathbf{R}^{2k+1}} (x_1 x_3 \cdots x_{2k-1}) \times \frac{\Delta(x_1, x_3, \dots, x_{2k-1})^2 \Delta(x_0, x_2, \dots, x_{2k})^2}{\prod_{i=1}^k \prod_{j=0}^k (x_{2i-1} + x_{2j})} d\mu^{2k+1}(x),$$

and then symmetrizing over all variables gives

$$D'_{k+1} = \frac{1}{k!(k+1)!(2k+1)!} \int_{\mathbf{R}^{2k+1}} \left(\sum_{\pi \in S_{2k+1}} (x_{\pi(1)} x_{\pi(3)} \cdots x_{\pi(2k-1)}) \times \frac{\Delta(x_{\pi(1)}, x_{\pi(3)}, \dots, x_{\pi(2k-1)})^2 \Delta(x_{\pi(0)}, x_{\pi(2)}, \dots, x_{\pi(2k)})^2}{\prod_{i=1}^k \prod_{j=0}^k (x_{\pi(2i-1)} + x_{\pi(2j)})} \right) d\mu^{2k+1}(x).$$

If $\sigma \in S_{2k+1}$ leaves the sets $\{0, 2, \dots, 2k\}$ and $\{1, 3, \dots, 2k-1\}$ invariant, then the terms corresponding to π and to $\pi \circ \sigma$ are equal, so each term actually appears $k!(k+1)!$ times in the sum above. Using the notation $\binom{[0, n]}{k}$ for the set of k -element subsets $\{i_1 < \dots < i_k\}$ of $\{0, 1, \dots, n\}$, as well as other notation introduced earlier in Definition 2.18, we can write D'_{k+1} in a less redundant way by picking out a “sorted representative” from each such group of equal terms:

$$\begin{aligned} D'_{k+1} &= \frac{1}{(2k+1)!} \int_{\mathbf{R}^{2k+1}} \left(\sum_{I, J} x_I \frac{\Delta_I^2 \Delta_J^2}{\Gamma_{I, J}} \right) d\mu^{2k+1}(x) \\ &= \frac{1}{(2k+1)!} \int_{\mathbf{R}^{2k+1}} \frac{1}{\Gamma_{I \cup J}} \left(\sum_{I, J} x_I \Delta_I^2 \Delta_J^2 \Gamma_I \Gamma_J \right) d\mu^{2k+1}(x), \end{aligned}$$

where the sum runs over all $\binom{2k+1}{k}$ ways of partitioning $\{0, 1, \dots, 2k\} = I \cup J$ into disjoint sets $I \in \binom{[0, 2k]}{k}$ and $J \in \binom{[0, 2k]}{k+1}$. Now we claim that the symmetric polynomial given by the sum in parentheses satisfies the identity

$$\sum_{I, J} x_I \Delta_I^2 \Delta_J^2 \Gamma_I \Gamma_J = \frac{1}{2^k} \sum_{I, J} \Delta_I^2 \Delta_J^2 \Gamma_{I, J}$$

with I and J running over the same sets as before. The general strategy for proving identities of this type is outlined in Appendix B, which also contains other identities needed in the proof of the remaining two statements (4.30) and

(4.32) of the present theorem. Granted the claim, we obtain

$$\begin{aligned}
D'_{k+1} &= \frac{2^{-k}}{(2k+1)!} \int_{\mathbf{R}^{2k+1}} \left(\sum_{I,J} \frac{\Delta_I^2 \Delta_J^2}{\Gamma_I \Gamma_J} \right) d\mu^{2k+1}(x) \\
&= \frac{2^{-k} k! (k+1)!}{(2k+1)!} \sum_{I,J} \left(\frac{1}{k!} \int_{\mathbf{R}^k} \frac{\Delta_I^2}{\Gamma_I} d\mu^k(x_{i_1}, \dots, x_{i_k}) \right. \\
&\quad \left. \times \frac{1}{(k+1)!} \int_{\mathbf{R}^{k+1}} \frac{\Delta_J^2}{\Gamma_J} d\mu^{k+1}(x_{j_1}, \dots, x_{j_{k+1}}) \right) \\
&= \frac{2^{-k}}{\binom{2k+1}{k}} \sum_{I,J} u_k u_{k+1} = \frac{u_k u_{k+1}}{2^k},
\end{aligned}$$

which proves (4.33). \square

Here we need yet some more notation. Let $\tilde{V}_{-1} = 0$, $\tilde{U}_0 = \tilde{V}_0 = 1$, and for $k \geq 1$:

$$\tilde{U}_k = \sum_{I \in \binom{[0,n]}{k}} \frac{(\Delta_I)^2}{\Gamma_I} b_I, \quad \tilde{V}_k = \sum_{I \in \binom{[0,n]}{k}} \frac{(\Delta_I)^2}{\Gamma_I} \lambda_I b_I, \quad (4.34)$$

In other words, \tilde{U}_k and \tilde{V}_k are symmetric functions of the same form as U_k and V_k of Definition 2.18, but involving the additional variables λ_0 and b_0 . For future use we also define, for $k \geq 0$,

$$\widetilde{W}_k = \begin{vmatrix} \tilde{U}_k & \tilde{V}_{k-1} \\ \tilde{U}_{k+1} & \tilde{V}_k \end{vmatrix}. \quad (4.35)$$

Lemma 4.11. *When $\mu = \sum_{i=0}^n b_i \delta_{\lambda_i}$, the integrals u_k and v_k in (4.28) reduce to the sums \tilde{U}_k and \tilde{V}_k , respectively. Hence, in this case,*

$$D_k = \frac{(\tilde{U}_k)^2}{2^k}, \quad D'_k = \frac{\tilde{U}_k \tilde{U}_{k-1}}{2^{k-1}}, \quad D''_k = \frac{\tilde{U}_k \tilde{V}_{k-1}}{2^{k-1}}. \quad (4.36)$$

Proof. This follows from the definitions and Lemma 4.10. \square

Recall that in the context of the discrete cubic string we have $\lambda_0 = 0$ and $b_0 = 1$ by definition.

Lemma 4.12. *When $\lambda_0 = 0$ and $b_0 = 1$,*

$$\tilde{U}_k = U_k + V_{k-1}, \quad \tilde{V}_k = V_k, \quad \widetilde{W}_k = W_k. \quad (4.37)$$

Proof. In the sum defining \tilde{U}_k , the terms for which $0 \notin I$ add up to U_k , while if $I = \{0\} \cup J$ with $J \in \binom{[1,n]}{k-1}$, then $(\Delta_I)^2/\Gamma_I = \lambda_J(\Delta_J)^2/\Gamma_J$ and $b_I = b_J$, so those terms add up to V_{k-1} . For \tilde{V}_k , the terms with $0 \in I$ vanish, leaving only terms which add up to V_k . Finally,

$$\widetilde{W}_k = \begin{vmatrix} U_k + V_{k-1} & V_{k-1} \\ U_{k+1} + V_k & V_k \end{vmatrix} = \begin{vmatrix} U_k & V_{k-1} \\ U_{k+1} & V_k \end{vmatrix} = W_k.$$

\square

4.4 Solution of the inverse problem

Now we merely have to put the pieces together.

Theorem 4.13. *The approximation problem of Definition 4.5 has a unique solution:*

1. *The coefficients of the polynomial $Q(z) = \sum_{i=0}^{k-1} q_i z^i$ are uniquely determined by the nonsingular linear system (4.21). In particular, the highest and lowest coefficients of $Q(z)$ are given by the following ratios of determinants:*

$$(-1)^{k-1} q_{k-1} = \frac{D'_k}{D_k}, \quad q_0 = \frac{D''_k}{D_k}. \quad (4.38)$$

2. *The polynomials $P(z)$ and $\hat{P}(z)$ are uniquely determined by $Q(z)$ as*

$$P = 1 + T_W Q, \quad \hat{P} = T_Z Q. \quad (4.39)$$

Proof. The determinant of (4.21) is $D_k = (\tilde{U}_k)^2 / 2^k = (U_k + V_{k-1})^2 / 2^k$, by the lemmas in the previous section. This is clearly positive since $U_k > 0$ and $V_{k-1} > 0$, so the system is nonsingular. Solving (4.21) for q_0 and q_{k-1} using Cramer's rule, one obtains the $k \times k$ determinants in Lemma 4.10, the factor $(-1)^{k-1}$ appearing when moving the β_i 's from the last column to the first in the case of q_{k-1} . Equation (4.39) is just a reformulation of (4.19). \square

Expressing the result in terms of \tilde{U} 's and \tilde{V} 's using (4.36) yields

Corollary 4.14.

$$(-1)^{k-1} q_{k-1} = \frac{2 \tilde{U}_{k-1}}{\tilde{U}_k}, \quad q_0 = \frac{2 \tilde{V}_{k-1}}{\tilde{U}_k}. \quad (4.40)$$

Remark 4.15. Note that there is a factor of \tilde{U}_k which cancels in the quotients (4.38). As should be clear from the proof of Lemma 4.10, this is a considerable complication compared to the classical case (Stieltjes continued fractions, Padé approximation, ordinary discrete string, Camassa–Holm peakons).

Returning to the discrete cubic string, recall that $(Q, P, \hat{P}) = (a_{12}, a_{22}, a_{32})$ is a solution of our approximation problem. Comparing equation (4.3) for $a_{12}(z)$ to equation (4.40) for $Q(z)$, we see that

$$\Omega_k := l_{k'} \left(\prod_{i=k'+1}^n \frac{g_i l_i^2}{2} \right) = \frac{2 \tilde{U}_{k-1}}{\tilde{U}_k} = \frac{2 (U_{k-1} + V_{k-2})}{U_k + V_{k-1}}. \quad (4.41)$$

Given the quantities in (4.41) one can easily compute the original string variables from

$$y_{k'} = 1 - \sum_{i=k'}^n l_i, \quad l_{k'} = y_{(k-1)'} - y_{k'}, \quad g_{k'} = \frac{2\Omega_{k+1}}{l_{(k+1)'} l_{k'} \Omega_k},$$

with the following result:

Theorem 4.16. *The solution to the inverse spectral problem for the discrete cubic string in terms of symmetric functions of λ 's and b 's is*

$$\begin{aligned} y_{k'} &= 1 - \frac{2\tilde{V}_{k-1}}{\tilde{U}_k} = \frac{U_k - V_{k-1}}{U_k + V_{k-1}} \quad (k = 1, \dots, n), \\ l_{k'} &= \frac{2\tilde{W}_{k-1}}{\tilde{U}_{k-1}\tilde{U}_k} = \frac{2W_{k-1}}{(U_{k-1} + V_{k-2})(U_k + V_{k-1})} \quad (k = 1, \dots, n+1), \\ g_{k'} &= \frac{(\tilde{U}_k)^4}{2\tilde{W}_{k-1}\tilde{W}_k} = \frac{(U_k + V_{k-1})^4}{2W_{k-1}W_k} \quad (k = 1, \dots, n), \end{aligned} \quad (4.42)$$

with U_k , V_k , and W_k as in Definition 2.18.

Remark 4.17. The formula for $y_{k'}$ is valid also for $k = 0$ and $k = n+1$; since $V_0 = U_{n+1} = 0$, it yields $y_{n+1} = +1$ and $y_0 = -1$, respectively, as it should.

Now we can finally prove the remaining part of Theorem 2.22, namely the explicit formulas for the inverse spectral problem in terms of peakon variables x_k and m_k .

Lemma 4.18.

$$x_{k'} = \log \frac{U_k}{V_{k-1}}, \quad m_{k'} = \frac{(U_k)^2 (V_{k-1})^2}{W_k W_{k-1}} \quad (k = 1, \dots, n). \quad (4.43)$$

Proof. By (3.5) the string variables can be expressed in terms of peakon variables as

$$x_{k'} = \log \frac{1 + y_{k'}}{1 - y_{k'}}, \quad m_{k'} = \frac{1}{8} g_{k'} (1 - y_{k'}^2)^2, \quad (4.44)$$

which immediately yields (4.43) upon inserting (4.42). \square

So far the solution of the inverse problem has been obtained under the assumption that the spectral measure is coming from a cubic string. Theorem 2.22 implies however that there is a bijection between discrete cubic strings and spectral measures μ .

Theorem 4.19. *For every discrete cubic string specified by a positive measure $g(y) = \sum_{i=1}^n g_i \delta_{y_i}(y)$ whose support satisfies $-1 < y_1 < y_2 < \dots < y_n < 1$ there exists a unique positive measure*

$$\mu = \sum_{i=0}^n b_i \delta_{\lambda_i}, \quad (4.45)$$

$b_0 = 1$, $b_k > 0$, $0 = \lambda_0 < \lambda_1 < \dots < \lambda_n$, such that the Weyl function $W(z)$ of the cubic string satisfies

$$\frac{W(z)}{z} = \int \frac{d\mu(\lambda)}{z - \lambda}.$$

Conversely, given a measure μ as in (4.45) there exists a unique cubic string defined by Theorem 4.16.

5 Acknowledgments

The research leading up to this work was initiated while the first author was visiting the Department of Mathematics and Statistics of the University of Saskatchewan, Canada, in the academic year 2002/2003. The research of the first author has been supported in part by the Department of Mathematics and Statistics, University of Saskatchewan, and by the National Science and Engineering Research Council of Canada (NSERC). The research of the second author is supported by NSERC. Both authors would like to thank NSERC and the Department of Mathematics and Statistics of the University of Saskatchewan for making this collaboration possible.

A Appendix: The discrete string

Here we give a brief account of the solution of the inverse spectral problem for a discrete string with Dirichlet boundary condition, for comparison to the discrete *cubic* string treated in this paper. For more details, we refer the reader to [2], and to Stieltjes' famous *Recherches sur les fraction continues*, especially the introduction and Chapter 3 [18, Vol. II]. It was M. G. Krein who pointed out the mechanical interpretation of Stieltjes' work and generalized it in his spectral theory of the general (not necessarily discrete) inhomogenous string [13]; see also Supplement II in [12], or the comprehensive treatment in [9].

Small vibrations $u(y, t)$ of a string with mass density $g(y)$ are described by the wave equation $u_{yy} = g(y)u_{tt}$. If the ends of the string are attached at $y = \pm 1$, separation of variables $u(y, t) = \phi(y)\tau(t)$ results in the following equation for $\phi(y)$:

$$\begin{aligned} \phi_{yy}(y) &= z g(y) \phi(y) \quad \text{for } y \in (-1, 1), \\ \phi(-1) &= 0, \quad \phi(1) = 0. \end{aligned} \tag{A.1}$$

This is a selfadjoint spectral problem with simple, negative (if $g > 0$) eigenvalues $z = \lambda_k = -\omega_k^2$, where ω_k is the frequency of the k th harmonic. The *discrete* string consisting of point masses g_i at positions $-1 < y_1 < \dots < y_n < +1$ corresponds to $g(y) = \sum_1^n g_i \delta(y - y_i)$ being a discrete measure. In this case, there are exactly n eigenvalues, and the eigenfunctions $\phi(y)$ are piecewise linear (as opposed to the well known sinusoidal shape in the homogeneous case $g(y) \equiv 1$). The piecewise constant slope $\phi_y(y)$ jumps by $z g_i \phi(y_i)$ at each point y_i .

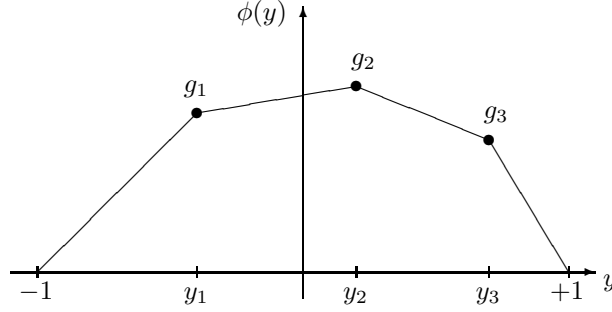


Figure 1: Discrete Dirichlet string with three masses

The spectral problem for the discrete string can be conveniently reformulated using the transition matrix associated with the initial value problem posed at the left end $y = -1$. One starts at $\phi(-1) = 0$ with initial slope $\phi_y(-1) = 1$, and propagates the solution to the right end point, which results in

$$\begin{pmatrix} \phi(1; z) \\ \phi_y(1; z) \end{pmatrix} = L_n G_n(z) \cdots L_1 G_1(z) L_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (\text{A.2})$$

where

$$L_k = \begin{pmatrix} 1 & l_k \\ 0 & 1 \end{pmatrix}, \quad G_k(z) = \begin{pmatrix} 1 & 0 \\ -zg_k & 1 \end{pmatrix}.$$

(Like for the cubic string, we set $l_k = y_{k+1} - y_k$, and also $k' = n + 1 - k$.) The eigenvalues are the zeros of the n th degree polynomial $\phi(1; z)$, and consequently they are the poles of the *Weyl function* of the problem, defined by

$$W(z) = \frac{\phi_y(1; z)}{\phi(1; z)}. \quad (\text{A.3})$$

With

$$\begin{aligned} L_n &= \begin{pmatrix} Q_0 & Q_1 \\ P_0 & P_1 \end{pmatrix} =: a^{(1)}, \\ L_n G_n(z) &= \begin{pmatrix} Q_2 & Q_1 \\ P_2 & P_1 \end{pmatrix} =: a^{(2)}, \\ L_n G_n(z) L_{n-1} &= \begin{pmatrix} Q_2 & Q_3 \\ P_2 & P_3 \end{pmatrix} =: a^{(3)}, \end{aligned} \quad (\text{A.4})$$

and so on, equation (A.2) is just a matrix version of the standard recurrence for the *convergents*

$$\frac{0}{1} = \frac{P_0}{Q_0}, \quad \frac{1}{l_n} = \frac{P_1}{Q_1}, \quad \frac{1}{l_n + \frac{1}{zg_n}} = \frac{P_2}{Q_2}, \quad \frac{1}{l_n + \frac{1}{zg_n + \frac{1}{l_{n-1}}}} = \frac{P_3}{Q_3}, \quad \dots$$

of a continued fraction expansion of Stieltjes type [18],

$$W(z) = \frac{1}{l_n + \frac{1}{zg_n + \frac{1}{l_{n-1} + \frac{1}{\ddots + \frac{1}{zg_2 + \frac{1}{l_1 + \frac{1}{zg_1 + \frac{1}{l_0}}}}}}} = \frac{P_{n+1}}{Q_{n+1}}. \quad (\text{A.5})$$

The *remainders*

$$R_0 = \frac{1}{l_0}, \quad R_1 = \frac{1}{zg_1 + \frac{1}{l_0}}, \quad R_2 = \frac{1}{l_1 + \frac{1}{zg_1 + \frac{1}{l_0}}}, \quad \dots$$

in the continued fraction are given by

$$R_{2k-1} = \frac{q_k}{p_k}, \quad R_{2k} = \frac{p_k}{q_{k+1}}, \quad \text{where} \quad \begin{pmatrix} q_k \\ p_k \end{pmatrix} = \begin{pmatrix} \phi(y_{k'}; z) \\ \phi_y(y_{k'}; z) \end{pmatrix}. \quad (\text{A.6})$$

Note that the convergents (P_j, Q_j) are obtained by formally setting the remainder R_j to zero in (A.5). Write

$$\frac{W(z)}{z} = \sum_{k=0}^n \frac{a_k}{z - \lambda_k} = \sum_{i=0}^{\infty} \frac{(-1)^j A_j}{z^{j+1}}, \quad (\text{A.7})$$

where $\lambda_0 = 0$, $a_0 = 1/2$, and where

$$A_j = \sum_{k=0}^n (-\lambda_k)^j a_j = \int x^j d\mu(x) \quad (\text{A.8})$$

is the j th moment of the *spectral measure*

$$\mu = \sum_{k=0}^n a_k \delta_{-\lambda_k}. \quad (\text{A.9})$$

The convergents are Padé approximants of W :

$$W(z) - \frac{P_{2k-1}(z)}{Q_{2k-1}(z)} = O\left(\frac{1}{z^{2k-1}}\right),$$

$$\deg P_{2k-1} = k - 1, \quad \deg Q_{2k-1} = k - 1, \quad P_{2k-1}(0) = 1,$$

in the odd case, and

$$W(z) - \frac{P_{2k}(z)}{Q_{2k}(z)} = O\left(\frac{1}{z^{2k}}\right),$$

$$\deg P_{2k} = k - 1, \quad \deg Q_{2k} = k, \quad Q_{2k}(0) = 1,$$

in the even case. These conditions uniquely determine the coefficients of $Q_j(z)$ in terms of certain Hankel determinants Δ_k^i involving the moments A_j appearing in the Laurent series of $W(z)$. Since

$$Q_{2k-1}(z) = (g_n l_n g_{n-1} l_{n-1} \cdots g_{k'}) z^{k-1} + \dots,$$

$$Q_{2k}(z) = (g_n l_n g_{n-1} l_{n-1} \cdots g_{k'} l_{k'}) z^k + \dots,$$

we can express the string data as ratios of coefficients of Q_j 's, which themselves are ratios of Hankel determinants, resulting in altogether four determinants. In the notation of [2],

$$l_{k'} = \frac{(\Delta_{k-1}^1)^2}{\Delta_{k-1}^0 \Delta_k^0}, \quad g_{k'} = \frac{(\Delta_k^0)^2}{\Delta_{k-1}^1 \Delta_k^1}.$$

These Hankel determinants can be evaluated in terms of λ_k 's and a_k 's using Heine's formula (4.26), thereby recovering the string data $\{g_i, l_i\}$ in terms of symmetric functions of the spectral data $\{\lambda_k, a_k\}$ encoded in the Weyl function.

B Appendix: Combinatorial identities

In the proof of Lemma 4.10 we used some combinatorial identities for symmetric polynomials.

Lemma B.1. *With the notation of Lemma 4.10 as well as that of Definition 2.18 in force, the following identities hold.*

1. *With the sums running over all partitions of $\{1, 2, \dots, 2k\}$ into disjoint subsets I and J with $|I| = |J| = k$,*

$$\sum_{I,J} \Delta_I^2 \Delta_J^2 \Gamma_I \Gamma_J x_I = \frac{1}{2^k} \sum_{I,J} \Delta_I^2 \Delta_J^2 \Gamma_{I,J}. \quad (\text{B.1})$$

2. *With the sums running over all partitions of $\{0, 1, 2, \dots, 2k\}$ into disjoint subsets I and J with $|I| = k$ and $|J| = k + 1$,*

$$\sum_{I,J} \Delta_I^2 \Delta_J^2 \Gamma_I \Gamma_J x_I = \frac{1}{2^k} \sum_{I,J} \Delta_I^2 \Delta_J^2 \Gamma_{I,J}, \quad (\text{B.2})$$

$$\sum_{I,J} \Delta_I^2 \Delta_J^2 \Gamma_I \Gamma_J (x_I)^2 = \frac{1}{2^k} \sum_{I,J} \Delta_I^2 \Delta_J^2 x_I \Gamma_{I,J}. \quad (\text{B.3})$$

Proof. We give a detailed proof only for the first identity. Almost identical technique can be used to verify the other identities. We start by noting that both sides of (B.1) are symmetric polynomials in the variables x_1, x_2, \dots, x_{2k} . In each single variable x_a both sides are of degree $3k - 2$. Let us denote the difference of the two sides by P . We claim that

1. P is divisible by $\Delta(x_1, x_2, x_3, \dots, x_{2k})$.
2. P is divisible by $\Delta^2(x_1, x_2, x_3, \dots, x_{2k})$.

Granted Claim 2 we easily conclude that $P = 0$ and hence the identity is proven. Indeed, the degree of P in, say, the x_1 variable is $3k - 2$, while that of $\Delta^2(x_1, x_2, x_3, \dots, x_{2k})$ is $4k - 2$.

Furthermore, Claim 2 follows from Claim 1 due to the symmetry of P , the skew-symmetry of $\Delta(x_1, x_2, x_3, \dots, x_{2k})$, and the fact that any polynomial skew-symmetric with respect to all transpositions of variables (in particular, then, the polynomial P/Δ) must be divisible by $\Delta(x_1, x_2, x_3, \dots, x_{2k})$.

Thus it suffices to prove Claim 1. Moreover, by symmetry, it suffices to show that P is divisible by $x_1 - x_2$. The proof proceeds by induction. The base case $k = 0$ is just $1 = 1$. Assume the identity holds for $k = m - 1$. We will be evaluating both sides of (B.1) at $x_1 = x_2$. In view of this, the only nonvanishing contributions to the left-hand side are coming from sets I and J such that $I = \{1\} \cup I'$ and $J = \{2\} \cup J'$, or such that $I = \{2\} \cup I''$ and $J = \{1\} \cup J''$. Thus the only terms contributing to the left-hand side will be

$$\begin{aligned} & \sum_{I', J'} \Delta_{\{1\}, I'}^2 \Delta_{I'}^2 \Delta_{\{2\}, J'}^2 \Delta_{J'}^2 \Gamma_{\{1\}, I'} \Gamma_{I'} \Gamma_{\{2\}, J'} \Gamma_{J'} x_1 x_{I'} \\ & + \sum_{I'', J''} \Delta_{\{2\}, I''}^2 \Delta_{I''}^2 \Delta_{\{1\}, J''}^2 \Delta_{J''}^2 \Gamma_{\{2\}, I''} \Gamma_{I''} \Gamma_{\{1\}, J''} \Gamma_{J''} x_2 x_{I''}. \end{aligned}$$

Since $I' \cup J' = I'' \cup J'' = \{3, 4, \dots, 2k\} =: A$, we obtain after evaluating at $x_1 = x_2$ that the above sums contribute

$$2x_2 \Delta_{\{2\}, A}^2 \Gamma_{\{2\}, A} \sum_{I', J'} \Delta_{I'}^2 \Delta_{J'}^2 \Gamma_{I'} \Gamma_{J'} x_{I'}.$$

Likewise, following the same steps, the right-hand side contributes

$$\frac{4x_2}{2^k} \Delta_{\{2\}, A}^2 \Gamma_{\{2\}, A} \sum_{I', J'} \Delta_{I'}^2 \Delta_{J'}^2 \Gamma_{I', J'}.$$

Since $|I'| = |J'| = k - 1$, the induction hypothesis ensures that the two contributions are equal. Thus P vanishes whenever $x_1 = x_2$, so it is divisible by $x_1 - x_2$. This proves Claim 1. \square

Remark B.2. The only similar identity for symmetric polynomials that we have come across in the literature is one involving $\Delta(x)^2$, but not $\Gamma(x)$, in a paper by Anderson [1] on the Toda lattice.

References

- [1] Arlen Anderson. An elegant solution of the n -body Toda problem. *J. Math. Phys.*, 37(3):1349–1355, 1996.
- [2] R. Beals, D. Sattinger, and J. Szmigielski. Multipeakons and the classical moment problem. *Advances in Mathematics*, 154:229–257, 2000.
- [3] R. Beals, D. H. Sattinger, and J. Szmigielski. Multi-peakons and a theorem of Stieltjes. *Inverse Problems*, 15(1):L1–L4, 1999.
- [4] Roberto Camassa and Darryl D. Holm. An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.*, 71(11):1661–1664, 1993.
- [5] A. Degasperis, D. D. Holm, and A. N. W. Hone. A new integrable equation with peakon solutions. *Theoretical and Mathematical Physics*, 133:1463–1474, 2002, nlin.SI/0205023.
- [6] A. Degasperis, D. D. Holm, and A. N. W. Hone. Integrable and non-integrable equations with peakons. In M. J. Ablowitz, M. Boiti, F. Pempinelli, and B. Prinari, editors, *Nonlinear Physics: Theory and Experiment (Gallipoli, 2002)*, volume II, pages 37–43. World Scientific, 2003, nlin.SI/0209008.
- [7] A. Degasperis and M. Procesi. Asymptotic integrability. In A. Degasperis and G. Gaeta, editors, *Symmetry and perturbation theory (Rome, 1998)*, pages 23–37. World Scientific Publishing, River Edge, NJ, 1999.
- [8] P. A. Deift. *Orthogonal polynomials and random matrices: a Riemann-Hilbert approach*, volume 3 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 1999.
- [9] H. Dym and H. P. McKean. *Gaussian processes, function theory, and the inverse spectral problem*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1976. Probability and Mathematical Statistics, Vol. 31.
- [10] Shaun M. Fallat, Michael I. Gekhtman, and Charles R. Johnson. Spectral structures of irreducible totally nonnegative matrices. *SIAM J. Matrix Anal. Appl.*, 22(2):627–645 (electronic), 2000.
- [11] Sergey Fomin and Andrei Zelevinsky. Total positivity: tests and parametrizations. *Math. Intelligencer*, 22(1):23–33, 2000.
- [12] F. P. Gantmacher and M. G. Krein. *Oscillation matrices and kernels and small vibrations of mechanical systems*. AMS Chelsea Publishing, Providence, RI, revised edition, 2002. Translation based on the 1941 Russian original, edited and with a preface by Alex Eremenko.

- [13] I. S. Kac and M. G. Krein. On the spectral functions of the string. *Amer. Math. Soc. Transl.*, 103(2):19–102, 1974.
- [14] M. Krein. Sur les fonctions de Green non-symétriques oscillatoires des opérateurs différentiels ordinaires. *C. R. (Doklady) Acad. Sci. URSS (N. S.)*, 25:643–646, 1939.
- [15] Hans Lundmark and Jacek Szmigielski. Multi-peakon solutions of the Degasperis–Procesi equation. *Inverse Problems*, 19:1241–1245, December 2003, nlin.SI/0503033.
- [16] Jürgen Moser. Finitely many mass points on the line under the influence of an exponential potential—an integrable system. In *Dynamical systems, theory and applications (Rencontres, Battelle Res. Inst., Seattle, Wash., 1974)*, pages 467–497. Lecture Notes in Phys., Vol. 38. Springer, Berlin, 1975.
- [17] E. M. Nikishin and V. N. Sorokin. *Rational approximations and orthogonality*, volume 92 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1991. Translated from the Russian by Ralph P. Boas.
- [18] Thomas Jan Stieltjes. *Œuvres complètes/Collected papers. Vol. I, II*. Springer-Verlag, Berlin, 1993.
- [19] G. Szegő. *Orthogonal polynomials*, volume XXIII of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1975.